Exam Number One
Administered: Thursday, October 14, 1999

Problem (1)

Consider the system of nonlinear algebraic equations:

$$x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2} = 4$$
$$x_3 = x_1^{\ 2} + x_2^{\ 2}$$
$$x_2 = \ln(x_1)$$

(a) How many solutions do you expect this equation to have?
(b) If we are going to solve this system using multivariate Newton-Raphson, we need the Jacobian and the residual. Determine them.
(c) For an initial guess of $(x_1, x_2, x_3) = (1,1,1)$, Evaluate the Jacobian and residual.
(d) Write the equation for the change in the unknowns and the new value of the unknowns at the next iteration.

Solution:
(a)

There is no general rule for the number of solutions to a system of nonlinear algebraic equations. There could be none or there could be an infinite number.
(b)

$$\underline{\underline{J}} = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2x_1 & 2x_2 & -1 \\ \frac{1}{x_1} & -1 & 0 \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2} - 4 \\ x_1^{\ 2} + x_2^{\ 2} - x_3 \\ \ln(x_1) - x_2 \end{bmatrix}$$

(c) For an initial guess of $(x_1, x_2, x_3) = (1,1,1)$, Evaluate the Jacobian and residual.

$$\underline{\underline{J}} = \begin{bmatrix} 2 & 2 & 2 \\ 2_1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

(d) Write the equation for the change in the unknowns and the new value of the unknowns at the next iteration.

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R}$$

$$\underline{x}^{j+1} = \underline{x}^{j} + \underline{\delta x}$$

Problem (2)

Say you are solving a system of nonlinear algebraic equations, using a numerical techniques, for example syseqn.m on MATLAB. You put in an initial guess for the unknowns and the program either crashes or doesn't converge. You are sure that there are no bugs in the program.

How do you proceed in solving the problem? What are you options? What are the possible sources of error?

Solution:

Possible sources of error:

(1) bad initial guess – (initial guess far from root, or initial guess in an unphysical region of phase space)

(2) derivative (gradients) near zero

Ways to proceed:

(1) Try a new random initial guess

(2) Use your understanding of the physical problem that gave rise to the equations to find a better initial guess

(3) Try a different numerical method. The simplex method (while slow) may converge where the Newton-Raphson method cannot.

(4) Solve a similar or asymptotic problem that is easier to solve. Then vary one parameter, solving for the solution at each variation, to slowly lead you back to the problem of interest.

Problem (3)

Consider the system of nonlinear ordinary differential equations:

$$\frac{dx_1}{dt} = x_1^2 + x_2^2 + x_3^2 - 4$$
$$\frac{dx_2}{dt} = x_1^2 + x_2^2 - x_3$$
$$\frac{dx_3}{dt} = \ln(x_1) - x_2$$

with the initial conditions $(x_1, x_2, x_3) = (1,1,1)$ at $t = 0$.

Perform one Euler step integration with a time step of $\Delta t = 0.1$.

Solution:

$$x_1^{j+1} = x_1^j + \Delta t \frac{dx_1}{dt} = x_1^j + \Delta t\left(x_1^2 + x_2^2 + x_3^2 - 4\right) = 1 + 0.1(1 + 1 + 1 - 4) = 0.9$$

$$x_2^{j+1} = x_2^j + \Delta t \frac{dx_2}{dt} = x_2^j + \Delta t\left(x_1^2 + x_2^2 - x_3\right) = 1 + 0.1(1 + 1 - 1) = 1.1$$

$$x_3^{j+1} = x_3^j + \Delta t \frac{dx_3}{dt} = x_3^j + \Delta t\left(\ln(x_1) - x_2\right) = 1 + 0.1(0 - 1) = 0.9$$

Problem (4)

Say you are solving a system of nonlinear ODEs, using a numerical techniques, for example sysode.m on MATLAB. You enter the initial conditions and the program crashes. You are sure that there are no bugs in the program.

How do you proceed in solving the problem? What are you options? What are the possible sources of error?

Solution:

Source of error:

(1) The time step is too large.

Ways to proceed:

(1) Decrease the time step.
(2) Use a higher order method

Problem (5)

If the determinant of an nxn matrix $\underline{\underline{A}}$ is zero, what can you say about:

(a) the rank of the matrix
(b) the linear dependence of the equations which form the matrix
(c) the inverse of the matrix
(d) the eigenvalues of the matrix
(e) the eigenvectors of the matrix
(f) the number of solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$

Solution:

(a) the rank of the matrix is less than n
(b) the equations which form the matrix are linearly dependent
(c) the inverse of the matrix does not exist
(d) At least one of the eigenvalues of the matrix is zero
(e) Can't say anything about the eigenvectors of the matrix
(f) the number of solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$ is infinite

Problem (6)

If you are trying to solve a system of linear algebraic equations and it turns out that the determinant is zero, but you still want a solution, what do you do? Give the steps in the algorithm.

Solution:

(1) Calculate determinant of the nxn matrix
(2) If determinant is zero, calculate rank.
(3) The number of independent equations is the rank. The number of arbitrary variables is $n - rank$.
(4) Arbitrarily select values for $n - rank$ arbitrary variables.
(5) Substitute the values for the $n - rank$ arbitrary variables into the n equations. Move all constant terms over to the right hand side to create a new $\underline{b}$ vector.
(6) Select $rank$ of the equations. You now have a $rank$ x $rank$ matrix.
(7) Calculate determinant of the $rank$ x $rank$ matrix.
(8) If determinant is zero, you picked a bad collection of equations. Go back to (6) and choose some other ones.
(9) If determinant is nonzero, solve your new $\underline{\underline{A}}\underline{x} = \underline{b}$ for the non-arbitrary variables.