ChE/MSE 505
Advanced Mathematic for Engineers
Final Exam
Fall Semester, 2003
Instructor: David Keffer
Administered: 8:00-10:00 am, Monday December 8, 2001

Consider the integral equation

$$\phi(x) = f(x) + \lambda\left[\int_{x_o}^{x} N(x,y)\phi(y)dy\right]$$

where

$$f(x) = x^2$$
$$N(x,y) = x(y+1)$$
$$\lambda = \frac{1}{2}$$
$$x_o = 1$$

(a) Is this integral equation linear or nonlinear?
(b) Is this integral equation Volterra or Fredholm?
(c) Is this integral equation of the first or second kind?
(d) Use a numerical method to find an approximate solution to $\phi(x)$ from x_o to $x_{f.}$=3. Use a discretization step of $\Delta x = 1$. You are free to solve this as you choose, as long as you state your assumptions. However, I suggest you use the trapezoidal rule to approximate the integral, although that is not mandatory. I would like to see numerical values for the solution. There is no use for calculators in this problem.

Solution:
Since the range of interest is 3-1=2 and the step size is 1, we will have n=2 intervals and n+1=3 points where the function is to be evaluated.

We write out the integral equation for each value of x=1, 2, and 3.

$$x = 1: \qquad \phi_1 = 1^2 + \frac{1}{2}\left[\int_1^1 1(y+1)\phi(y)dy\right]$$

$$x = 2: \qquad \phi_2 = 2^2 + \frac{1}{2}\left[\int_1^2 2(y+1)\phi(y)dy\right]$$

$$x = 3: \qquad \phi_3 = 3^2 + \frac{1}{2}\left[\int_1^3 3(y+1)\phi(y)dy\right]$$

We use the Trapezoidal rule to evaluate the integral

$$\int_{x_o}^{x_f} f(y)dy = \frac{\Delta x}{2}\left[f(x_o) + f(x_f) + 2\sum_{j=2}^{n} f(x_j)\right]$$

In the first equation, the integral is zero because the upper and lower limits of integration are the same. The equations become.

$$x = 1: \qquad \phi_1 = 1$$

$$x = 2: \qquad \phi_2 = 4 + \frac{1}{2}\left[(1+1)\phi_1 + (2+1)\phi_2\right] = 4 + \phi_1 + \frac{3}{2}\phi_2$$

$$x = 3: \qquad \phi_3 = 9 + \frac{3}{2}\frac{1}{2}\left[(1+1)\phi_1 + (3+1)\phi_3 + 2(2+1)\phi_2\right] = 9 + \frac{3}{2}\phi_1 + \frac{9}{2}\phi_2 + 3\phi_3$$

This is a set of two linear algebraic equations. There are only two unknowns because equation (1) provides the value of ϕ_1. Now we rewrite the equations:

$$x = 2: \qquad -\frac{1}{2}\phi_2 = 4 + \phi_1 = 5$$

$$x = 3: \qquad -\frac{9}{2}\phi_2 - 2\phi_3 = 9 + \frac{3}{2}\phi_1 = 9 + \frac{3}{2}1 = \frac{21}{2}$$

Simplify a little more for the sake of convenience

$$x = 2: \qquad \phi_2 = -10$$

$$x = 3: \qquad 9\phi_2 + 4\phi_3 = -21$$

Solving the last equation for ϕ_3 yields

$$x = 3: \qquad \phi_3 = -\frac{1}{4}(21 + 9\phi_2) = -\frac{1}{4}(-69) = \frac{69}{4}$$

So the solution is approximated by:

$$x = 1: \qquad \phi_1 = 1$$

$$x = 2: \qquad \phi_2 = -10$$

$$x = 3: \qquad \phi_3 = \frac{69}{4}$$

This is probably a very bad approximation. We would require a much finer discretization to get a more accurate picture of the solution.