ChE/MSE 505
Advanced Mathematic for Engineers
Final Exam
Fall Semester, 2001
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Consider the integro-differential equation

$$c_0(x)\frac{d\phi(x)}{dx} + c_1(x)\phi(x) + c_2(x)\left[\int_{x_o}^{x_f} N(x,y)\phi(y)dy\right] + c_3(x) = 0$$

where

$$c_0(x) = 1$$
$$c_1(x) = -1$$
$$c_2(x) = -1$$
$$c_3(x) = -e^x + 1$$
$$N(x,y) = e^{x-y}$$
$$x_o = 0$$
$$x_f = 2$$

with the initial condition

$$\phi(x = x_o) = 1$$

(a) Characterize the equation as linear or nonlinear.
(b) Use a numerical method to find an approximate solution to $\phi(x)$ from x_o to x_f. Use a discretization step of $\Delta x = 1$. You are free to solve this as you choose, as long as you state your assumptions. However, I suggest you use a centered-finite difference formula to approximate the derivative at internal nodes and a backward-finite difference formula to approximate the derivative at the last node. Also, I suggest you use the trapezoidal rule to approximate the integral, although that too is not mandatory. I would like to see numerical values for the solution.

Solution:
Since the range of interest is 2 and the step size is 1, we will have n=2 intervals and n+1=3 points where the function is to be evaluated. Of these three points, the first is given by the initial condition. The solution will be of the form

$$\underline{\phi}^{all} = \begin{bmatrix} \phi(x = x_0 = 0) = \phi_o \\ \phi(x = x_1 = 1) \\ \phi(x = x_2 = 2) \end{bmatrix}$$

But only the last two of these are unknown. So it is more useful to write our vector of unknowns as only

$$\underline{\phi} = \begin{bmatrix} \phi(x = x_1 = 1) \\ \phi(x = x_2 = 2) \end{bmatrix}$$

We use a centered finite difference form to evaluate the derivative at all nodes except the first node, where we are forced to use a forward finite difference formula, and the last node, where we are forced to use a backward finite difference formula.

$$\left.\frac{d\phi}{dx}\right|_{x_i} = \begin{cases} \dfrac{\phi(x_{i+1}) - \phi(x_{i-1})}{2\Delta x} & \text{for } 1 \le i \le n \\ \dfrac{\phi(x_i) - \phi(x_{i-1})}{\Delta x} & \text{for } i = n+1 \end{cases}$$

where n is the number of intervals. We use the Trapezoidal rule to evaluate the integral

$$\int_{x_o}^{x_f} f(y)dy = \frac{\Delta x}{2}\left[f(x_o) + f(x_f) + 2\sum_{j=2}^{n} f(x_j) \right]$$

Substituting the finite difference rule and the Trapezoidal rule into the original equation for i=1 and i=2, yields

$$c_0(x_1)\frac{\phi(x_2) - \phi(x_o)}{2\Delta x} + c_1(x_1)\phi(x_1) + c_2(x_1)\frac{\Delta x}{2}\left[N(x_1, x_o)\phi(x_o) + N(x_1, x_2)\phi(x_2) + 2N(x_1, x_1)\phi(x_1)\right] + c_3(x_1) = 0$$

$$c_0(x_2)\frac{\phi(x_2) - \phi(x_1)}{\Delta x} + c_1(x_2)\phi(x_2) + c_2(x_2)\frac{\Delta x}{2}\left[N(x_2, x_o)\phi(x_o) + N(x_2, x_2)\phi(x_2) + 2N(x_2, x_1)\phi(x_1)\right] + c_3(x_2) = 0$$

This is a system of 2 linear algebraic equations, which can be expressed as

$$\underline{\underline{A}}\underline{\phi} = \underline{b}$$

where

$$\underline{\underline{A}} = \begin{bmatrix} c_1(x_1) + c_2(x_1)N(x_1, x_1)\Delta x & \frac{c_o(x_1)}{2\Delta x} + \frac{c_2(x_1)N(x_1, x_2)\Delta x}{2} \\ -\frac{c_o(x_2)}{\Delta x} + c_2(x_2)N(x_2, x_1)\Delta x & \frac{c_o(x_2)}{\Delta x} + c_1(x_2) + \frac{c_2(x_2)N(x_2, x_2)\Delta x}{2} \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} -c_3(x_1) + c_0(x_1)\frac{\phi(x_o)}{2\Delta x} - c_2(x_1)\frac{\Delta x}{2}N(x_1, x_o)\phi(x_o) \\ -c_3(x_2) - c_2(x_2)\frac{\Delta x}{2}N(x_2, x_o)\phi(x_o) \end{bmatrix}$$

The solution is given by

$$\underline{\phi} = \underline{\underline{A}}^{-1}\underline{b}$$

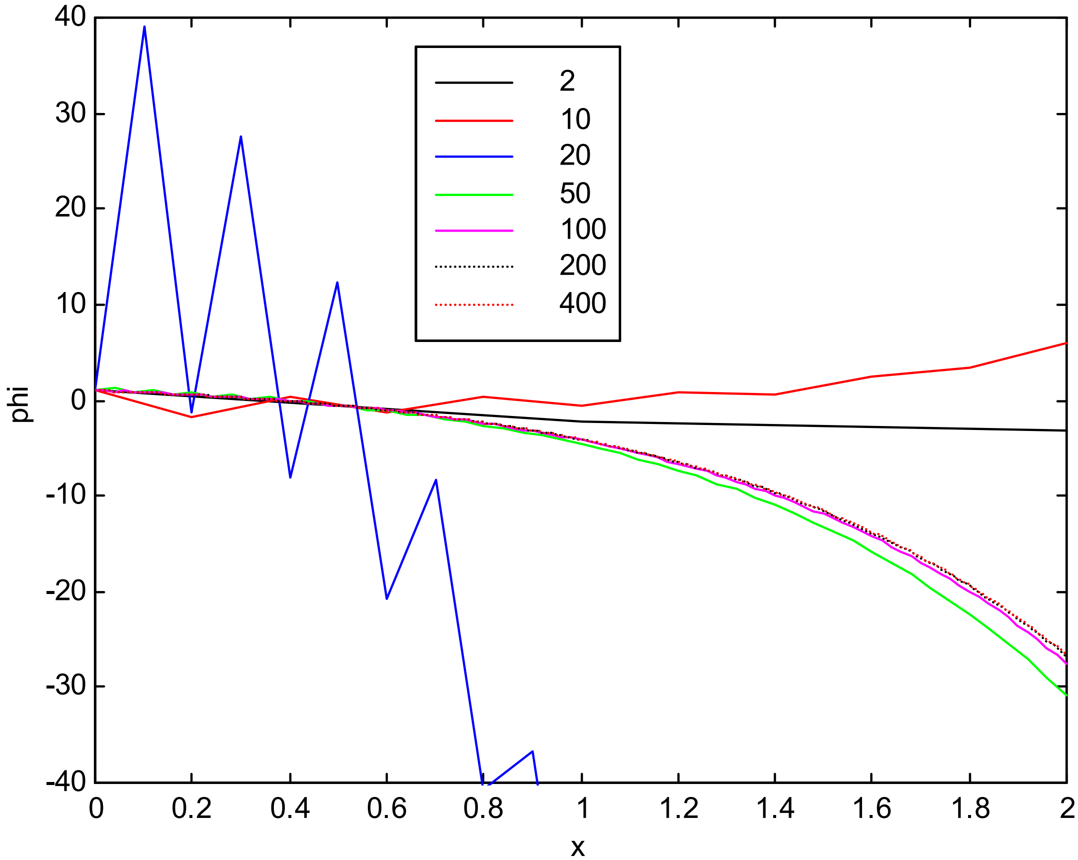
Let's numerically evaluate the matrix A and vector b

i	x	c0	c1	c2	c3	N(x,0)	N(x,1)	N(x,2)
0	0	1	-1	-1	0	1	0.367879	0.135335
1	1	1	-1	-1	-1.71828	2.718282	1	0.367879
2	2	1	-1	-1	-6.38906	7.389056	2.718282	1

A matrix		b vector	A inverse		phi
-2	0.31606	3.577423	-0.22986	-0.1453	-2.28748
-3.71828	-0.5	10.08358	1.709397	-0.91946	-3.15617

det(A) = 2.175201

The solution is plotted below for several different numbers of intervals. You can see that it takes 100 or so intervals to really get a good approximation of the solution.



40
30
20
10
0
-10
-20
-30
-40
phi
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x
2
10
20
50
100
200
400