Midterm Examination
Administered: Wednesday, October 10, 2001

Problem (1)

Consider the 2x2 matrix:

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

(a) Find the eigenvalues.
(b) Find the normalized eigenvectors.
(c) If $a_{11} = a_{22}$, does $\lambda_1 = \lambda_2$? Why or why not?
(d) If $a_{11} = a_{22}$, does $\underline{w}_1 = \underline{w}_2$? Why or why not?

Solution:

(a) Find the eigenvalues.

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = \det\begin{bmatrix} a_{11}-\lambda & 0 \\ 0 & a_{22}-\lambda \end{bmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) = 0$$

Therefore, $\lambda_1 = a_{11}$ and $\lambda_2 = a_{22}$

(b) Find the normalized eigenvectors.

$$\left(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}\right)\underline{w}_1 = \underline{0}$$

$$\begin{bmatrix} a_{11}-a_{11} & 0 \\ 0 & a_{22}-a_{11} \end{bmatrix}\underline{w}_1 = \begin{bmatrix} 0 & 0 \\ 0 & a_{22}-a_{11} \end{bmatrix}\underline{w}_1 = \underline{0}$$

Rank of $\left(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}\right)$ is 1 by inspection. Also, solution by inspection is $\underline{w}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The second element must be zero to satisfy the equation. The first element can be selected arbitrarily. Since the eigenvector is to be normalized, we select a value of one.

An entirely analogous process, leads to a second eigenvector, $\underline{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) If $a_{11} = a_{22}$, does $\lambda_1 = \lambda_2$? Why or why not?

Yes, the eigenvalues are the same, because $\lambda_1 = a_{11} = \lambda_2 = a_{22}$ as was derived in part (a).

(d) If $a_{11} = a_{22}$, does $\underline{w}_1 = \underline{w}_2$? Why or why not?

No. The eigenvectors would be the same as they were in part (b) because the matrix $(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ has a rank of 0, so both values of the eigenvectors can be selected arbitrarily. If we want orthonormal eigenvectors, then we can stick with those from part (b).

Problem (2)

Consider the system of nonlinear algebraic equations:

$$\sin(x_1) + x_2 = 0$$
$$x_1^2 + x_2^2 = 1$$

where x_1 has units of radians.

(a) If we are going to solve this system using multivariate Newton-Raphson, we need the Jacobian and the residual. Determine them.

(b) For an initial guess of $(x_1, x_2) = (\frac{1}{2}, -\frac{1}{2})$, Evaluate the Jacobian and residual.

(c) What is the next estimate of the solution using multivariate Newton-Raphson?

solution:

(a) If we are going to solve this system using multivariate Newton-Raphson, we need the Jacobian and the residual. Determine them.

$$\underline{\underline{J}} = \begin{bmatrix} \cos(x_1) & 1 \\ 2x_1 & 2x_2 \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} \sin(x_1) + x_2 \\ x_1^2 + x_2^2 - 1 \end{bmatrix}$$

(b) For an initial guess of $(x_1, x_2) = (\frac{1}{2}, -\frac{1}{2})$, Evaluate the Jacobian and residual.

$$\underline{\underline{J}} = \begin{bmatrix} 0.8776 & 1 \\ 1 & -1 \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} -0.0206 \\ -0.5 \end{bmatrix}$$

(c) What is the next estimate of the solution?

$$\det(\underline{\underline{J}}) = -0.8776 - 1 = -1.8776$$

$$\underline{\underline{J}}^{-1} = \frac{1}{\det(\underline{\underline{J}})} \begin{bmatrix} -1 & -1 \\ -1 & 0.8776 \end{bmatrix} = \begin{bmatrix} 0.5326 & 0.5326 \\ 0.5326 & -0.4674 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{\underline{J}}^{-1}\underline{R} = -\begin{bmatrix} 0.5326 & 0.5326 \\ 0.5326 & -0.4674 \end{bmatrix} \begin{bmatrix} -0.0206 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0.2773 \\ -0.2227 \end{bmatrix}$$

$$\underline{x}^{j+1} = \underline{x}^{j} + \underline{\delta x} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} + \begin{bmatrix} 0.2773 \\ -0.2227 \end{bmatrix} = \begin{bmatrix} 0.7773 \\ -0.7227 \end{bmatrix}$$

Problem (3)

If the determinant of an nxn matrix $\underline{\underline{A}}$ is 1.0, what can you say about:

(a) the rank of the matrix
(b) the linear dependence of the equations which form the matrix
(c) the inverse of the matrix
(d) the eigenvalues of the matrix
(e) the eigenvectors of the matrix
(f) the number of solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$

solution:

(a) the rank of the matrix

The rank of the matrix is n

(b) the linear dependence of the equations which form the matrix

All equation represent by A are independent

(c) the inverse of the matrix

The inverse of A exists

(d) the eigenvalues of the matrix

The eigenvalues of A are all non-zero.

(e) the eigenvectors of the matrix

Can't say much about the eigenvectors.

(f) the number of solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$

There is one unique solution to this system of equations.

Problem (4)

Consider the ordinary differential equation:

$$\frac{d^3y}{dx^3} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$$

where f is in general a nonlinear function, and the following conditions:

$$y(x = x_o) = y_o \qquad \left.\frac{dy}{dx}\right|_{x=x_o} = y'_o \qquad \left.\frac{dy}{dx}\right|_{x=x_f} = y'_f$$

Provide an algorithm for numerically obtaining a solution to this ordinary differential equation. Explain any transformations or recastings of the equations necessary. Name and describe particular numerical methods required in the solution.

solution:

Recast the third-order ODE as three first order ODEs. Recognize that this problem is a BVP. Solve as an IVP using the shooting method in conjunction with a Runge-Kutta method. This means you must guess y''_o and solve the system of 3 first order ODEs out to time x_f. At time x_f, evaluate $\left.\frac{dy}{dx}\right|_{x=x_f}$. If this value matches the boundary condition within an acceptable tolerance, you are done. Otherwise, pick a new value of y''_o and try again. The linear interpolation formula for $\left.\frac{dy}{dx}\right|_{x=x_f}$ as a function of y''_o can still be used.