Understanding Fluid Flow by Modeling Efflux from a Tank

A Unit Operations Laboratory Experiment

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I. Project Objectives

The purpose of this experiment is to demonstrate the predictive capabilities of the Bernoulli Equation in determining the time it takes a liquid to drain, under the influence of gravity, from a tank, through an exit pipe, as a function of initial tank charge, exit pipe diameter, and exit pipe length.

This project is comprised of an experimental component and a modeling component. In the modeling component, predictions of the efflux time are obtained from several different approximate solutions of the Bernoulli equation. In the experimental component, the efflux time for water draining from a tank through various exit pipes is measured.

Comparisons between the empirical and theoretical values are then made. The purposes of the comparison are (1) to evaluate which terms of the Bernoulli equation are important and (2) to test the limits of applicability of the Bernoulli equation.

II. Experimental System

Our system is situated inside a cylindrical tank (tank radius = R_T) filled with water to height, H. The tank has a cylindrical pipe (pipe radius = R_P) of length L extending from the base of the tank.

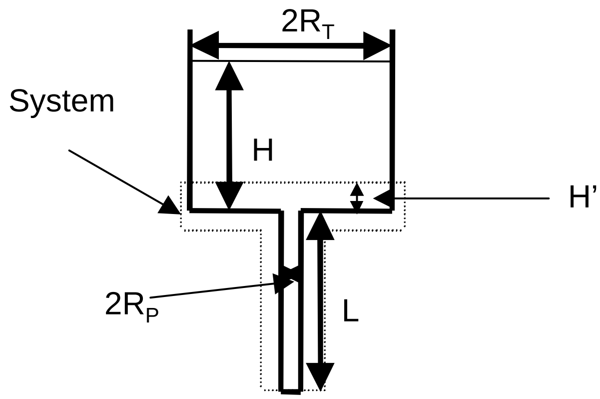

Figure One. Efflux from a Tank

The length and the diameter of the exit pipe are variables depending upon which of the eight available pipes is used. The pipes are made of stainless steel. The pipe dimensions are given in Table 1.

Table 1
Pipe Dimensions

Length (inches)	Inside Diameter (inches)
30	3/16
24	3/16
12	3/16
6	3/16
1	3/16
24	1/8
24	1/4
24	5/16

III. Mathematical Model

The mathematical model of the system is derived by first considering the mass and mechanical energy balances. The system around which the balance equations are written is given by the dotted line and extends a height H' into the tank.

Mass Balance

$$\text{accumulation} = \text{in} - \text{out} + \text{generation} \tag{1}$$

During the experiment, the system remains filled with water, so there is no accumulation term. There is, of course, no internal generation of water. There are only in and out terms. The in and out terms are the product of the velocity in the tank or pipe (v_T, v_P respectively) and the area of the tank or pipe (A_T, A_P respectively).

$$\text{in} = v_T A_T = v_T \pi R_T^2 \quad \text{and} \quad \text{out} = v_P A_P = v_P \pi R_P^2 \tag{2.a}$$

so the mass balance, upon rearrangement and simplification, becomes

$$v_T R_T^2 = v_P R_P^2 \tag{2.b}$$

The velocity of the tank is defined as

$$v_T = \frac{dH}{dt} \tag{3.a}$$

where t is time. Equation (3.a) can be substituted into equation (2.b) to yield an expression for the velocity in the pipe.

$$v_P = \frac{dH}{dt}\frac{R_T^2}{R_P^2} \tag{3.b}$$

Mechanical Energy Balance

The mechanical energy balance is drawn over the same system (defined by the dotted line in Figure One).

$$\frac{g\Delta z}{g_c} + \frac{\Delta v^2}{2g_c} + \frac{\Delta P}{\rho} + \sum h_f = 0 \tag{4}$$

where g is gravity, $\Delta z = L + H'$, $\Delta v^2 = v_T^2 - v_P^2$, ΔP is the pressure drop, ρ is the density of the fluid, and h_f are the terms contributing to the head loss due to friction.

Model One. Neglect kinetic energy term and frictional head losses due to flow in tank and contraction

It is reasonable to assume that

Assumption One. Pressure drop. The pressure at the top of the system (height of H') is given by

$$P_{H'} = P_{atm} + \frac{\rho g(H - H')}{g_c} \tag{5.a}$$

and the pressure at the bottom of the pipe is atmospheric pressure, so the total pressure drop is just the negative of the second term on the right hand side of equation (5.a)

Assumption Two. Kinetic Energy. Experience has taught us that in a situation like this, the velocity is negligible. (This makes the analysis easier since the velocity is a function of time. However, it is not a necessary assumption, when we have a tool like MATLAB at our disposal.)

Assumption Three. Friction Although there should be (at least) three terms in $\sum h_f$, one accounting for the friction in the tank walls, one accounting for the contraction where the tanks adjoins the pipe, and one accounting for the friction in the pipe walls, so that we would have

$$\sum h_f = h_{f,tankwall} + h_{f,contraction} + h_{f,pipewall} \tag{5.b}$$

we are going to ignore the first two terms of equation (5.b) and consider only the friction head loss due to the pipe wall. The Darcy equation gives the friction head loss for flow in a straight pipe:

$$h_{f,pipewall} = 4\left(\frac{fL}{D_P}\right)\frac{v_P^2}{2g_c} \tag{5.c}$$

where f is a dimensionless friction factor and D_P is the diameter of the pipe.

Making these three assumptions and substituting them into equation (4) yields:

$$-g(L + H) + \frac{fL2v_P^2}{D_P} = 0 \tag{5.d}$$

Empirical relation for the friction factor for turbulent flow

We can obtain an estimate of the friction factor, f, using an empirical relation, known as the Blasius equation, applicable to turbulent flow with Reynolds numbers in the range of 4000 < N_{Re} < 100,000.

$$f = \frac{0.0791}{N_{Re}^{0.25}} \tag{6.a}$$

where the Reynolds number in the pipe is defined as

$$N_{Re,P} = \frac{D_P \rho v_P}{\mu} \tag{6.b}$$

Using the Blasius equation results in two assumptions. First, we assume we have turbulent flow. Second, we assume that the pipe is smooth. (Actually the Blasius equation is the function plotted on the Moody charts (page 88, Geankoplis 3rd Ed. or page 5-24, Perry's 6th Ed.) when ε/D = 0.

We can substitute equations (6.a) and (6.b) into equation (5.d) to obtain

$$-g(L+H) + \frac{2(0.0791)\mu^{0.25} L v_P^{1.75}}{\rho^{0.25} D_P^{1.25}} = 0 \tag{7}$$

At this point, equation we still have two variables that are functions of time, H and v_P. By substituting our mass balance (equation (3.b)) into our mechanical energy balance (equation (5.d)), we eliminate v_P, arriving at

$$-g(L+H) + \frac{2(0.0791)\mu^{0.25} L D_T^{3.5}\left(\frac{dH}{dt}\right)^{1.75}}{\rho^{0.25} D_P^{4.75}} = 0 \tag{8}$$

We now have only one function of time, H. Equation (8) is a first order linear ordinary differential equation. Given an initial condition, namely $H(t=0) = H_o$, we can find a unique solution. It turns out for Case One, that the there is an analytical solution. To obtain the analytical solution, we can rearrange equation (8) to a suitable form

$$\left(1 + \frac{H}{L}\right)^{4/7} = -\left[\frac{2(0.0791)\mu^{0.25} D_T^{3.5}}{g\rho^{0.25} D_P^{4.75}}\right]^{4/7}\left(\frac{dH}{dt}\right) \tag{9}$$

Setting the term in brackets to be C and rearranging further, we have,

$$\int_{t'=0}^{t'=t} dt' = -C^{4/7} \int_{H''=H_o}^{H''=H} \left(1+\frac{H''}{L}\right)^{-4/7} dH'' \tag{10}$$

where H_o is the initial height of the water in the tank at time zero. Upon integration, we have

$$t = C^{4/7} L \frac{7}{3}\left[\left(1+\frac{H_o}{L}\right)^{3/7} - \left(1+\frac{H(t)}{L}\right)^{3/7}\right] \tag{11}$$

This equation concludes the derivation needed for parts 1. through 4. of the report requirements.

Model Two: Include kinetic energy term

If we choose not to neglect the kinetic energy term but continue to make the other assumptions, we arrive at a new version of equation (5.d)

$$-g(L+H) + \frac{fL2v_P^2}{D_P} + \frac{\Delta v^2}{2} = 0 \tag{12}$$

$$-g(L+H) + \frac{fL2v_P^2}{D_P} + \frac{v_P^2 - v_T^2}{2} = 0 \tag{13}$$

Now, just as we did in the previous case, we substitute the Blasius equation (6.a) and the definition of the Reynold's number (6.b) into the mechanical energy balance (because we are stil including the head loss due to flow in the pipe, which is non-negligible.

$$-g(L+H) + \frac{2(0.0791)\mu^{0.25} L v_P^{1.75}}{\rho^{0.25} D_P^{1.25}} + \frac{v_P^2 - v_T^2}{2} = 0 \tag{14}$$

At this point we recall that v_T is just the derivative of H with respect to time (equation (3.a)) and that the mass balance yielded a relation for v_P in terms of H (equation (3.b)). Substituting these two equations into equation (14) yields:

$$-g(L+H) + \frac{2(0.0791)\mu^{0.25} L \left(\frac{dH}{dt}\frac{D_T^2}{D_P^2}\right)^{1.75}}{\rho^{0.25} D_P^{1.25}} + \frac{\left(\frac{dH}{dt}\frac{D_T^2}{D_P^2}\right)^2 - \left(\frac{dH}{dt}\right)^2}{2} = 0 \tag{15}$$

Rearrangement yields:

$$-g(L+H)+\left(\frac{dH}{dt}\right)^{1.75}\left[\frac{2(0.0791)\mu^{0.25}LD_T^{3.5}}{\rho^{0.25}D_P^{4.75}}\right]+\left(\frac{dH}{dt}\right)^{2}\left[\frac{\left(\frac{D_T^4}{D_P^4}\right)-1}{2}\right]=0 \qquad (16)$$

This function describes a first-order non-linear ordinary differential equation. The initial condition needed to specify a unique solution is

$$H(t=0)=H_o \qquad (17)$$

Unlike equation (8) for case one, there is no analytical solution for Case Two (equation (16)). Therefore, we solve equation (16) given equation (17) using MATLAB or any other language or numerical software. Sample code that I wrote in MATLAB to solve equation (8) numerically is provided at http://clausius.engr.utk.edu/che310/index.html . This code can also be used for Case Two, with extremely simple modifications of the code.

Model Three: Account for the frictional head loss due to the contraction

If we return to Case One and decide to include the frictional head loss due to the contraction (but continue to ignore the kinetic energy term) then we need an expression for that head loss at the contraction. One formula used for such purposes is taken from the Crane Manual (Crane, Technical Paper No. 410, 18th Edition, New York, 1979) which says

$$h_{f,contraction}=K_c\,\frac{v_P^2}{2g_c} \qquad (18)$$

where K_c is called the resistance coefficient and is defined as (for our example)

$$K_c=0.5\left(1-\frac{D_P^2}{D_T^2}\right) \qquad (19.a)$$

If we look in Geankoplis, we find

$$K_c=0.55\left(1-\frac{D_P^2}{D_T^2}\right) \qquad (19.b)$$

Perry's handbook has other values for K_c, (page 5-34, 6th Ed.). We have to just choose of the definitions, so let's go with the Crane Manual.

If we include equation (18) and (19.a) in our mechanical energy balance, then we have

$$-g(L+H)+\frac{fL2v_P^2}{D_P}+\frac{1}{2}\left(1-\frac{D_P^2}{D_T^2}\right)\frac{v_P^2}{2}=0 \tag{20}$$

We make the same substitution for f, v_P, and v_T as we did in Case One and Case Two. These three substitutions yield

$$-g(L+H)+\frac{2(0.0791)\mu^{0.25}LD_T^{3.5}\left(\frac{dH}{dt}\right)^{1.75}}{\rho^{0.25}D_P^{4.75}}+\frac{1}{4}\left(1-\frac{D_P^2}{D_T^2}\right)\left[\frac{D_T^2}{D_P^2}\left(\frac{dH}{dt}\right)\right]^2=0 \tag{21}$$

This function describes a first-order non-linear ordinary differential equation. The initial condition needed to specify a unique solution is

$$H(t=0)=H_o \tag{17}$$

We can solve equation (21) given equation (17) using MATLAB or any other language or numerical software. The same code I used for Case One (available on the web) can also be used for Case Three, with extremely simple modifications of the code.

<u>Model Four: Account for the frictional head loss due to flow in the tank</u>

If we return to Case One and decide to include the frictional head loss due to the flow in the tank (but continue to ignore the kinetic energy term and the contraction term) then we need an expression for that head loss in the tank. The flow in the tank is laminar. You can verify this with your data. The head loss due to flow in the tank is given as

$$h_{f,tankwall}=4\left(\frac{fH}{D_T}\right)\frac{v_T^2}{2g_c} \tag{22}$$

For laminar flow, the friction factor is given as

$$f=\frac{16}{N_{Re}} \tag{23}$$

where the Reynolds number in the tank is defined as

$$N_{Re,T} = \frac{D_T \rho v_T}{\mu} \tag{24}$$

We can substitute equations (22), (23) and (24) into equation (4) to obtain

$$-g(L+H) + \frac{2(0.0791)\mu^{0.25} L D_T^{3.5} \left(\frac{dH}{dt}\right)^{1.75}}{\rho^{0.25} D_P^{4.75}} + \frac{32H\mu v_T}{D_T^2 \rho} = 0 \tag{25}$$

From the mass balance, we know v_T

$$-g(L+H) + \frac{2(0.0791)\mu^{0.25} L D_T^{3.5} \left(\frac{dH}{dt}\right)^{1.75}}{\rho^{0.25} D_P^{4.75}} + \frac{32H\mu}{D_T^2 \rho}\left(\frac{dH}{dT}\right) = 0 \tag{26}$$

This function describes a first-order linear ordinary differential equation. The initial condition needed to specify a unique solution is

$$H(t=0) = H_o \tag{17}$$

We can solve equation (21) given equation (17) using MATLAB or any other language or numerical software. The same code I used for Case One (available on the web) can also be used for Case Three, with extremely simple modifications of the code.

Of course, these are only four of an infinite number of possible models. It should be recognized that these four models were selected by the instructor for their diversity. Other models could be considered. For example, a model which doesn't assume smooth pipes could be used. Or, a model which simultaneously includes friction loss due to contraction, kinetic energy, and friction loss due to flow in the tank could be considered.

IV. Experimental Procedure

All four models described above require the same set of experiments.

A tank six inches in diameter with removable drain pipes is used to collect data on tank efflux times. A pipe is connected to the tank and the tank filled with water to a predetermined height. The tank is then drained. The stop watch is started at the initial height, H_o. The tank should be drained until one inch of water remains. The stop watch is stopped at the final time, H_f, when one inch of water remains. Three different values of H_o should be used for each pipe. Example values of H_o are 8.0, 13.0, and 20.0 cm, but these values are intended to be representative. This should be repeated for each of the eight pipes. (See Table 1.)

Several questions arise as to detailed procedures during the experiment. For example, should the flow of water be started from a standstill at H_o? Or, alternatively, should the flow be started at some point above H_o and allowed to reach fully-developed flow by the time the level reaches H_o, at which point the clock would be started. You need to think this through.

V. Report Requirements

This project has some elements which *must* be included in the report and some elements which *could or should* be included in the report.

The mandatory requirements include

- plot - experimental and model 1 theoretical (eq. 11) efflux times as a function of pipe diameter for the 24 inch long pipe
- plot - experimental and model 1 theoretical (eq. 11) efflux times as a function of pipe length for the 3/16 inch diameter pipe
- plot - experimental and model 1 theoretical (eq. 11) efflux times as a function of height of water for the pipe of length 24 inches and 3/16 inch diameter
- plot - experimental and model 2 theoretical (eq. 11) efflux times as a function of pipe diameter for the 24 inch long pipe
- plot - experimental and model 2 theoretical (eq. 11) efflux times as a function of pipe length for the 3/16 inch diameter pipe
- plot - experimental and model 3 theoretical (eq. 11) efflux times as a function of pipe diameter for the 24 inch long pipe
- plot - experimental and model 4 theoretical (eq. 11) efflux times as a function of pipe diameter for the 24 inch long pipe
- verify turbulent flow in the pipe

The mandatory discussions include

- determination of the relationship between efflux time and pipe length, as shown by the experimental data and theoretical models
- determination of the relationship between efflux time and pipe diameter, as shown by the experimental data and theoretical models
- determination of the relationship between efflux time and initial water height, as shown by the experimental data and theoretical models
- discussion of the agreement/disagreement between theory and experiment for the three relationships given above

The additional possibilities for the report include

- verify laminar flow in the tank
- discussion and explanation of when different terms of the Bernoulli equation are important, for example, "including the kinetic energy term, is more/less important for long drain pipes because…"
- possible sources of experimental error
- possible sources of error in the mathematical model