Stability Analysis of ODEs

David Keffer
Department of Chemical Engineering
University of Tennessee, Knoxville
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I. One Linear ODE with constant coefficient

We begin the study of the stability analysis of ordinary differential equations with the simplest example possible, namely a single linear ODE with constant coefficient. The ODE has the form

$$\frac{dy}{dx} = ay + b \tag{I.1}$$

subject to the initial condition $y(x = x_o) = y_o$. This ODE has an analytical solution, obtained through the method of the integrating factor, given by

$$y(x) = \left(y_o + \frac{b}{a} \right) \exp\left(a\left(x - x_o\right)\right) - \frac{b}{a} \tag{I.2}$$

In this form, it can be observed that the solution of the ODE has an exponential form. The stability of this system can be evaluated by examining the behavior of $y(x)$ in the limit as *x* approaches infinity

$$\lim_{x \to \infty} y(x) = \begin{cases} -\dfrac{b}{a} & if\ a < 0 \\ \infty & if\ a > 0 \end{cases} \tag{I.3}$$

The stability depends on the sign of *a*, the constant coefficient. When *a* is negative, the solution converges to a finite value. When *a* is positive, the solution diverges to infinity. In the language of stability analysis, we shall say that a converging solution exhibits stable behavior and a diverging solution exhibits unstable behavior.

It will be useful to identify *a* as the eigenvalue of this problem. In the subsequent problems, we shall see that the sign of the eigenvalues is critical in determining the stability of a system of ODEs.

What's the point of stability analysis? The point of stability analysis is to determine what sort of qualitative behavior you can expect from the solution to a system of ODEs, without actually having to solve them. In this trivial case, we can determine the stability of this ODE simply from the sign of *a*. While the systems that follow become more complicated, the over-riding purpose of stability analysis, namely predicting the qualitative behavior of the ODEs, remains the same. The value of stability analysis becomes more important as the systems become more complicated and our intuitive understanding of the expected behavior begins to fail us.

II. Two linear ODEs with constant coefficients

The classic example of stability analysis involves two linear ODEs with constant coefficients. This is a very limited subset of problems but it is worth seeing what rigorous criteria for stability we can achieve. We will proceed by examining a system of 2 linear ODEs. The work will apply for a system of n linear ODEs (as will be shown in the next section) but it is much easier to visualize in two dimensions.

We have a system of ODEs of the form

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} \tag{II.1}$$

and we have an initial condition of the form:

$$\underline{y}(x = x_o) = \underline{y}_o \tag{II.2}$$

This system of equations has a critical point at $\underline{y}_c = \underline{y}(x_c)$, where $\underline{y}_c$ satisfies the condition:

$$\frac{dy_2}{dy_1} = \frac{dy_2/dx}{dy_1/dx} = \frac{a_{21}y_1 + a_{22}y_2 + b_2}{a_{11}y_1 + a_{12}y_2 + b_1} = \frac{0}{0} \tag{II.3}$$

which yields

$$\underline{y}_c = -\underline{\underline{A}}^{-1}\underline{b} = \frac{1}{\det\left(\underline{\underline{A}}_{2x2}\right)}\begin{bmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{bmatrix} \tag{II.4}$$

where the determinant of the 2x2 matrix is $\det\left(\underline{\underline{A}}_{2x2}\right) = a_{11}a_{22} - a_{21}a_{12}$.

Clearly the critical points are the values of $\underline{y}$ when the derivatives of $\underline{y}$ are zero. However, when the derivatives are zero, the function is constant, so these critical points are often thought of as the steady-state, or equilibrium, or long-time (depending on the problem) solutions to the ODE. This physical analogy only applies, however, when the critical points are stable.

All trajectories pass through the critical point, including the eigenvectors.

Types of Critical Points

There are five types of critical points. The type of critical point depends on two and only two features of the ODEs: (1) the nature of the eigenvalues (real, imaginary or complex) and (2) the sign of the real component of the eigenvalues. The result is summarized in the following table.

type of critical point	nature of eigenvalues	stability
improper node	real	stable if all eigenvalues < 0
proper node	real	stable if all eigenvalues < 0
saddle point	real	unstable
center	imaginary	stable
spiral point	complex	stable if real component of all eigenvalues < 0

For a 2x2 matrix, the characteristic equation is

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0 \tag{II.5}$$

with eigenvalues given by the quadratic equation

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4\det\left(\underline{\underline{A}}_{2x2}\right)}}{2} \tag{II.6}$$

From here, we see that the eigenvalues are strictly a function of $\underline{\underline{A}}$ and can be real, purely imaginary conjugates or complex conjugates. Thus the nature and sign of these eigenvalues determine the type and stability of critical points.

Example II.1. Stable Improper Node

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

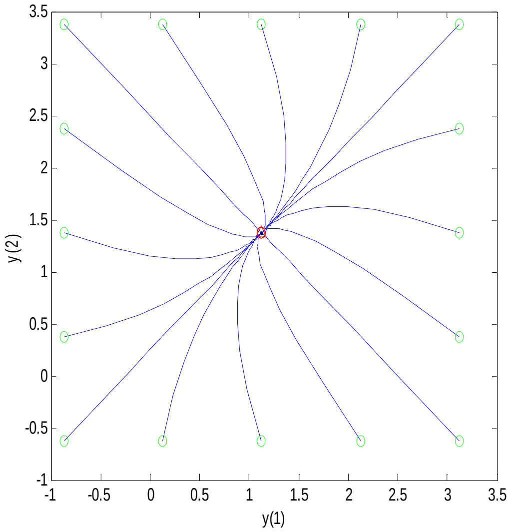
The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} 1.125 \\ 1.375 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

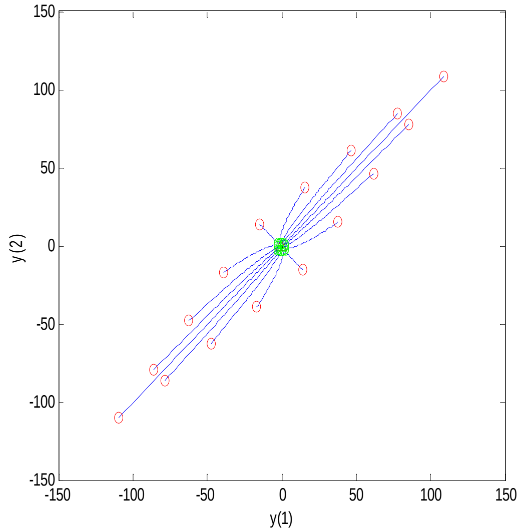
The eigenvalues are real and both negative. Therefore, the node is a stable node. It turns out to be an improper node (the most common type of node). We will distinguish from a proper node shortly.

We present a phase plot below. A phase plot is a plot of y_1 vs y_2. Critical points can be plotted on a phase plot. Trajectories (that is the solution of the ODEs from various initial conditions) can be parametrically plotted on a phase plot. In the following phase plot and all phase plots that follow, the trajectories are shown in blue. The initial conditions are shown in green. The final point of the trajectory is shown in red. As a reminder, each trajectory involves the solution (analytical or numerical) of the ODE with a different initial condition.

Because all of the trajectories in the phase plot for Example II.1. approach the critical point, we have visual evidence of its stability. The eigenvectors correspond to the only straight-line trajectories in the phase plot.



Phase Plot: Example II.1. Stable Improper Node.



Phase Plot: Example II.2. Unstable Improper Node.

Example II.2. Unstable Improper Node

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} -0.3755 \\ -0.875 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

The eigenvalues are real and both positive. Therefore, the node is an unstable node. It turns out to be an improper node. From the phase plot, it is clear that all trajectories diverge from the critical point.

Example II.3. Stable Proper Node

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues are real and both negative. Therefore, the node is a stable node. It is called proper because all trajectories are straight in the phase plot (because the ODEs are not coupled).

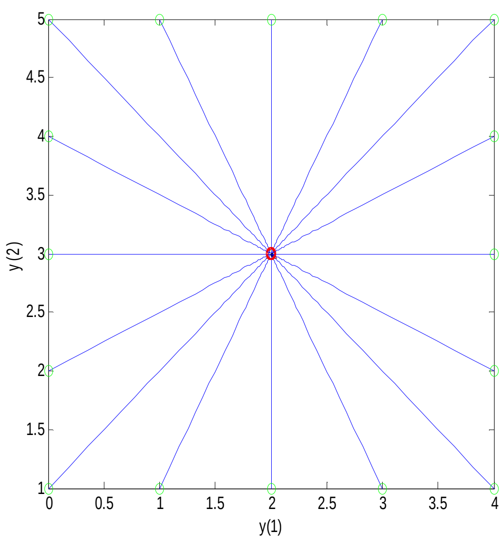
Example II.4. Unstable Proper Node

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

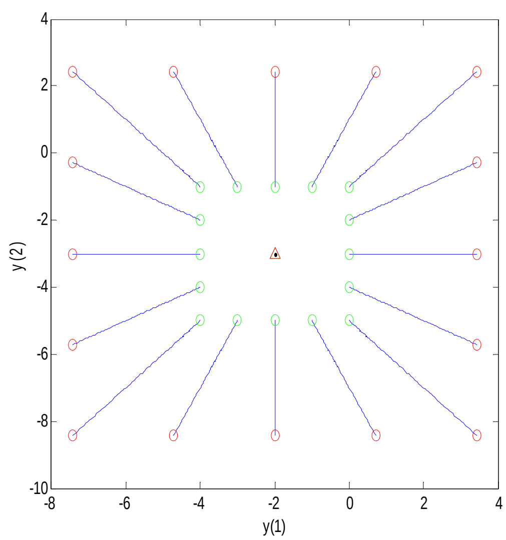
The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues are real and both positive. Therefore, the node is an unstable proper node.



Phase Plot: Example II.3. Stable Proper Node.



Phase Plot: Example II.4. Unstable Proper Node.

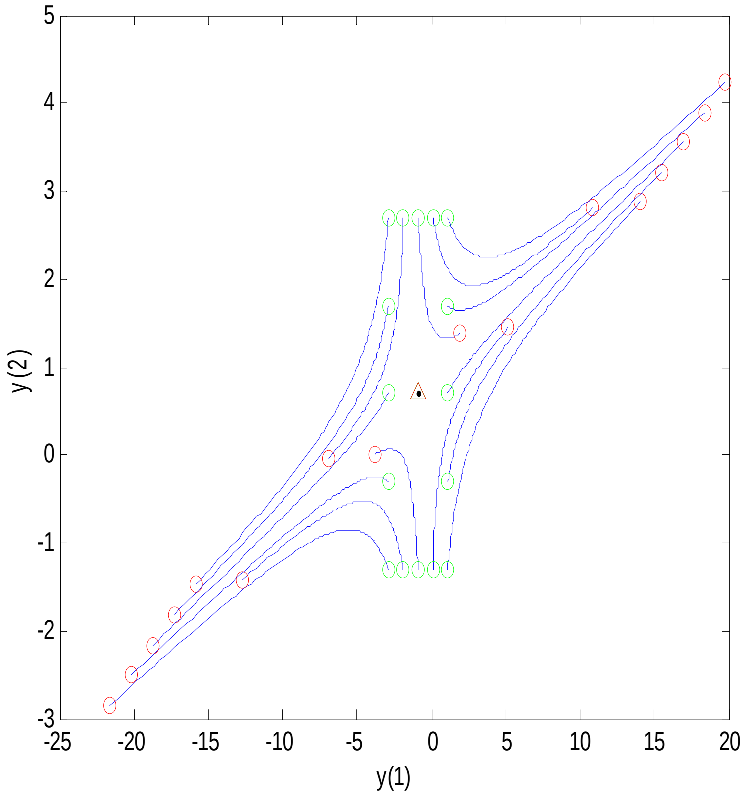
Example II.5. (Unstable) Saddle Point

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} -0.9 \\ 0.7 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -3.1623 & 0 \\ 0 & 3.1623 \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} 0.1602 & -0.9871 \\ -0.9871 & -0.1602 \end{bmatrix}$$

The eigenvalues are both real. One is positive the other is negative. Therefore, the node is a saddle point. Saddle points are by definition always unstable. The instability of saddle points stems from the fact that in order for a node to be a stable all eigenvalues must be negative. Since the signs of the eigenvalues of a saddle point are different, one of them must be positive and the critical point must be unstable.



Phase Plot: Example II.5. (Unstable) Saddle Point.

From the phase plot of the saddle point we observe that some trajectories initially approach the critical point, as they follow along the eigenvector associated with the negative eigenvalue. Eventually these trajectories diverge as they follow along the eigenvector associated with the positive eigenvalue.

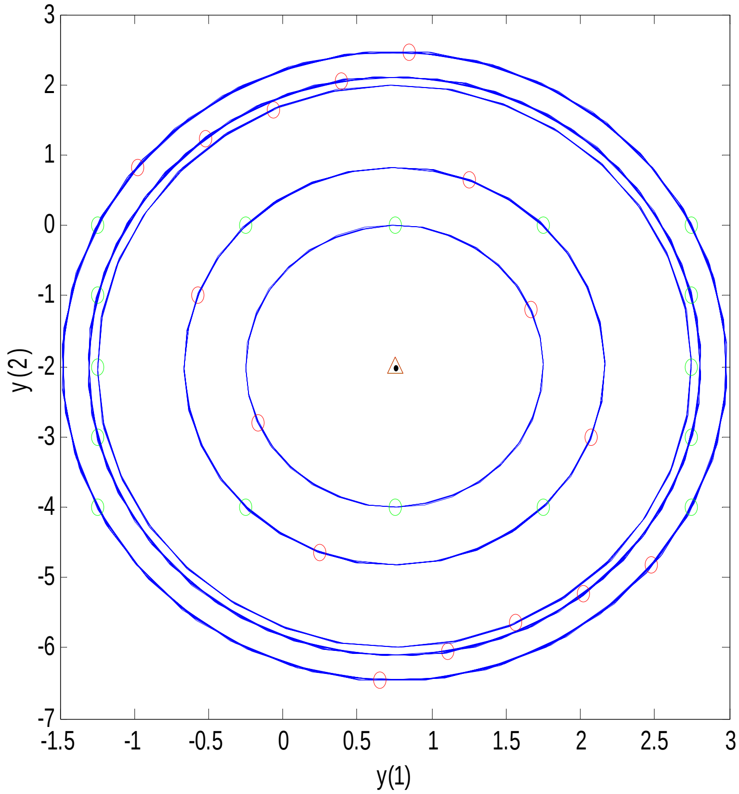
Example II.6. (Stable) Center

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} -0.75 \\ -2 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} -0.4472i & 0.4472i \\ 8.944 & 9.944 \end{bmatrix}$$

The eigenvalues are both purely imaginary. The real components of both eigenvalues are zero. Therefore, the critical point is center. Centers are by definition always stable.



Phase Plot: Example II.6. (Stable) Center.

From the phase plot of the center we observe that trajectories orbit the critical point. They neither converge nor diverge. The shape of the orbit need not be a circle nor an ellipse. For nonlinear systems, the shape of the orbit can be very unusual, but it is a stable orbit, if it is a center.

Example II.7. Stable Spiral Point

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2\, i & -\sqrt{2}/2\, i \end{bmatrix}$$

The eigenvalues are complex conjugate with negative real components. Therefore, the node is a stable spiral point.

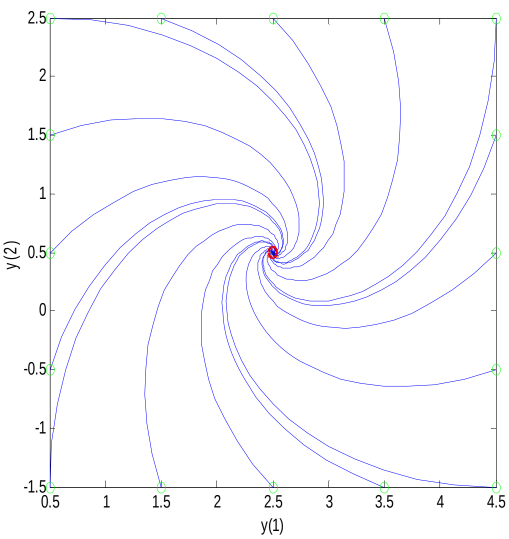
Example II.8. Unstable Spiral Point

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

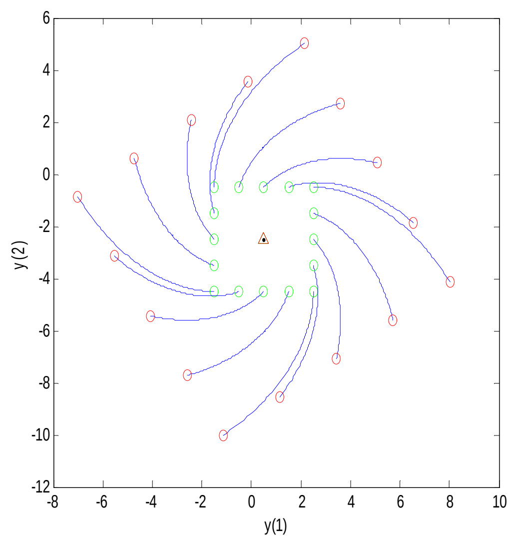
The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} 0.5 \\ -2.5 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2\, i & -\sqrt{2}/2\, i \end{bmatrix}$$

The eigenvalues are complex conjugate with positive real components. Therefore, the node is an unstable spiral point.



Phase Plot: Example II.7. Stable Spiral Point.



Phase Plot: Example II.8. Unstable Spiral Point.

III. N linear ODEs with constant coefficients

In this section, we extend the classical stability analysis for two linear ODEs with constant coefficients to a larger system of the same type of ODEs. We still have a system of ODEs of the form

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} \tag{II.1}$$

and we have an initial condition of the form:

$$\underline{y}(x = x_o) = \underline{y}_o \tag{II.2}$$

For a system of 2 ODEs, the definition of a critical point was

$$\frac{dy_2}{dy_1} = \frac{dy_2/dx}{dy_1/dx} = \frac{0}{0} \tag{II.3}$$

By analogy, the definition of a critical point for a system of *n* linear ODEs is

$$\frac{dy_i}{dx} = 0 \text{ for all } 1 \le i \le n \tag{III.1}$$

The critical point is then given by

$$\underline{y}_c = -\underline{\underline{A}}^{-1}\underline{b} \tag{III.2}$$

If the inverse exists, then the determinant is zero and we have a unique solution to this algebraic equation. *So that we see that a system of linear equations, regardless of the size, has only one critical point.* (Here we neglect cases where the system is poorly posed and there are infinite or no solutions to eqn (III.2).)

The determination of the type of critical point is based on the same criteria as that provided in the previous section. We have a node or a saddle point if all eigenvalues are real. We have a center if all eigenvalues are imaginary. We have a spiral point if all eigenvalues are complex. If we have some combination of real and complex eigenvalues, then we will have some combination of behaviors, as shall be demonstrated in the example below.

The stability criterion remains unchanged. The real components of all eigenvalues must be negative (or zero) for the critical point to be stable.

Example III.1. 3 linear ODEs with Constant Coefficients

Consider the system of linear ODEs:

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\,\underline{y} + \underline{b} \tag{II.1}$$

where

$$\underline{\underline{A}} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

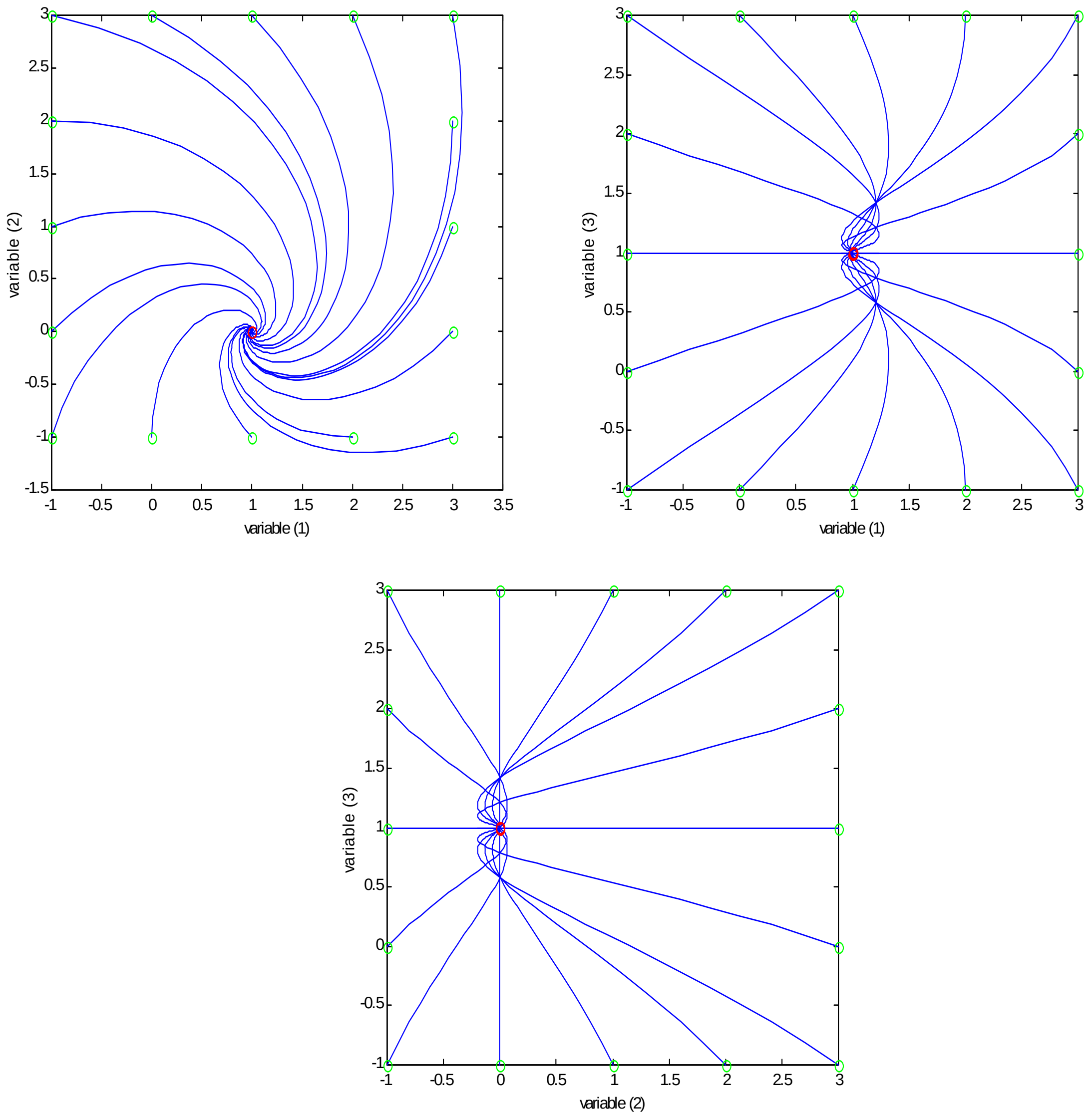
The critical point, eigenvalues and eigenvectors are

$$\underline{y}_c = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \underline{\underline{\Lambda}} = \begin{bmatrix} -1+i & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \underline{\underline{W}}_c = \begin{bmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All of the real parts of the eigenvalues are negative. Therefore, we should expect a stable critical point. Some of the eigenvalues have nonzero imaginary parts, therefore we should expect a spiral, at least in some dimensions of phase space.

For these higher-dimensional problems, two-dimensional phase plots can be generated by pairing any two variables. On the following page, all three combinations of variables are used to generate three phase plots.

In the phase plot featuring y_1 and y_2, we observe a spiral, as is consistent with the first two eigenvalues, which are complex conjugates. In the phase plot featuring y_1 and y_3, as well as y_2 and y_3, we observe some combination of a spiral and improper node, consistent with their respective eigenvalues.



Phase Plots: Example III.1. Stable point with features of a spiral point and improper node.

IV. Linear ODEs with variable coefficients

We now release the constraint of constant coefficients. In general, we can add functionality to both the $\underline{\underline{A}}$ matrix and the $\underline{b}$ vector.

$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}(x)\underline{y} + \underline{b}(x) \tag{IV.1}$$

and we have initial conditions of the form:

$$\underline{y}(x = x_o) = \underline{y}_o \tag{IV.2}$$

If either $\underline{\underline{A}}$ or $\underline{b}$ are functions of *x*, then the critical point will be a function of *x*. This means the critical points change in time (if *x* represents a temporal dimension). In other words, we now have a moving steady state. For the 2x2 case, we have an analytical solution for the critical points

$$\underline{y}_c(x) = -\underline{\underline{A}}^{-1}(x)\underline{b}(x) = \frac{1}{\det\left(\underline{\underline{A}}_{2x2}(x)\right)}\begin{bmatrix} a_{22}(x)b_1(x) - a_{12}(x)b_2(x) \\ -a_{21}(x)b_1(x) + a_{11}(x)b_2(x) \end{bmatrix} \tag{IV.3}$$

where the determinant of the 2x2 matrix is $\det\left(\underline{\underline{A}}_{2x2}(x)\right) = a_{11}(x)a_{22}(x) - a_{21}(x)a_{12}(x)$.

The eigenvalues and eigenvectors are functions of *x*, only if $\underline{\underline{A}}$ is a function of *x*. (The eigenvalues and eigenvectors are independent of $\underline{b}$.) At any instant in *x*, the criteria for types and stability of critical points holds. However, since the eigenvalues change with *x*, it is possible for the type and sign of the eigenvalue to change with *x*. Consequently, it is possible for the type of the critical point and/or the stability to change with *x*. If the type and stability hold for a range of *x*, then we can expect the associated behavior of the solution to hold over that range.

Example IV.1. 2 linear ODEs with Constant Coefficients but $\underline{b}=f(x)$

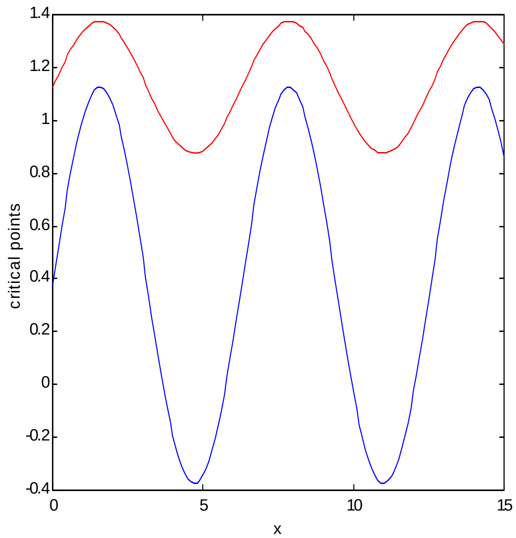
$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b}(x) = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\underline{y} + \begin{bmatrix} 2\sin(x) \\ 3 \end{bmatrix}$$

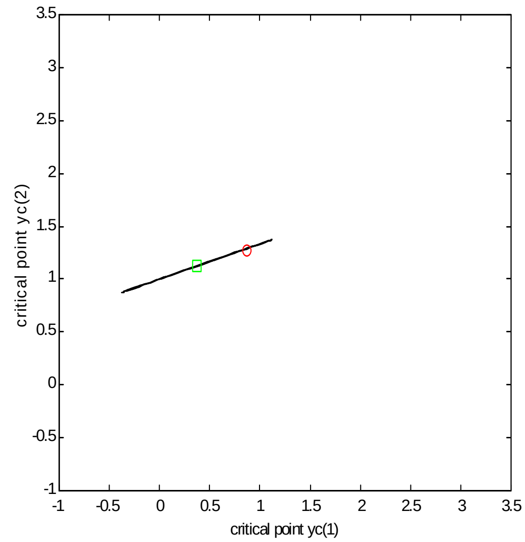
Since $\underline{\underline{A}}$ is constant, the eigenvalues and eigenvectors are constants

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \qquad \underline{\underline{W}}_c = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are real and negative, so we know to observe the behavior of a stable improper node.

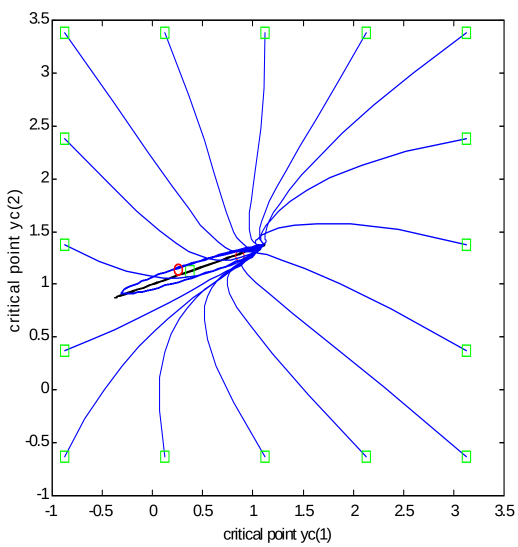
Using equation (IV.3) we find that the critical points as a function of *x* look like:





In the figure on the left, we have plotted the first component of the critical point, y_c, as a function of *x* in red and the second component of the critical point as a function of *x* in blue. In the figure on the right, we have plotted $y_{c,2}$ vs $y_{c,1}$ as parametric functions of *x*. (The figure on the right has the same axes as the phase plot.) Let's call this curve of critical points the critical path.

Below we show several solutions to the ODE, starting from different initial conditions. The starting points of each line are indicated by green squares. The ending points are indicated by red circles.



Phase Plots: Example IV.1. Stable improper node following moving steady state.

We see that the solution is indeed an attractor. All points lead to the critical path. Moreover, they lead to the critical path in a node-like way, without spiraling. The curious behavior is at the center where we do find a cyclical steady state. This cyclical steady state is due to the sine function in $\underline{b}(x)$. It is interesting that the trajectories never actually fall on the black line indicated by the critical path, but rather form a cycle about it. This must be due to the fact that the ODEs at time x are heading toward a solution defined by $\underline{b}(x)$. However, at some incremental time later, x', the solution has now moved and is defined by $\underline{b}(x')$. Thus, the path of the ODE must be altered. The solution can be said to lag behind the critical path. All solutions eventually find the same lag, as indicated by the fact that regardless of the initial condition, the final position (in this case plotted at x=15) is the same.

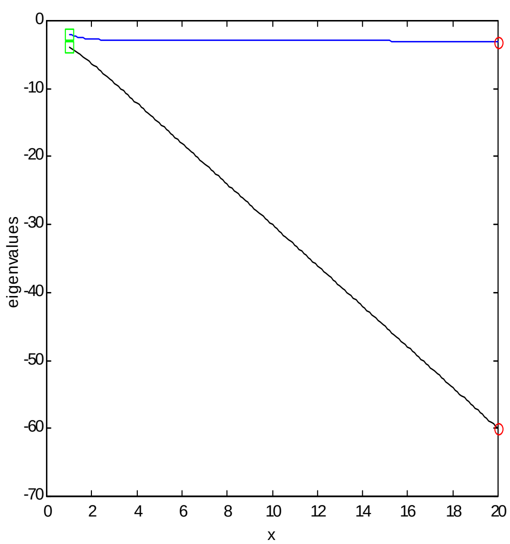
Example IV.1. 2 linear ODEs with Variable Coefficients

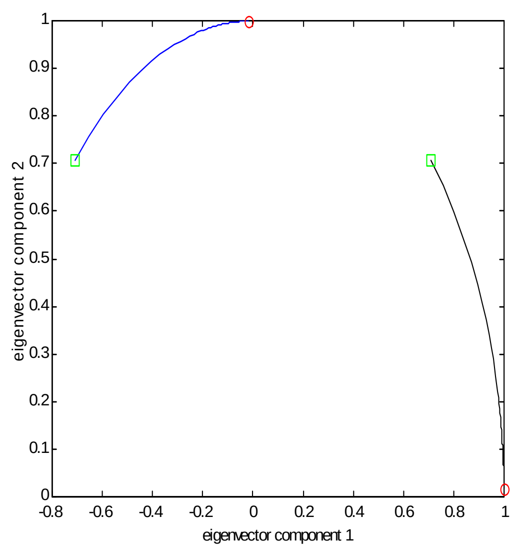
$$\frac{d\underline{y}}{dx} = \underline{\underline{A}}\underline{y} + \underline{b}(x) = \begin{bmatrix} -3x & 1 \\ 1 & -3 \end{bmatrix}\underline{y} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The eigenvalues and eigenvectors are now functions of x, given by

$$\lambda(x) = \frac{(a_{11}(x) + a_{22}(x)) \pm \sqrt{(a_{11}(x) + a_{22}(x))^2 - 4\det\left(\underline{\underline{A}}_{2x2}(x)\right)}}{2} \tag{IV.4}$$

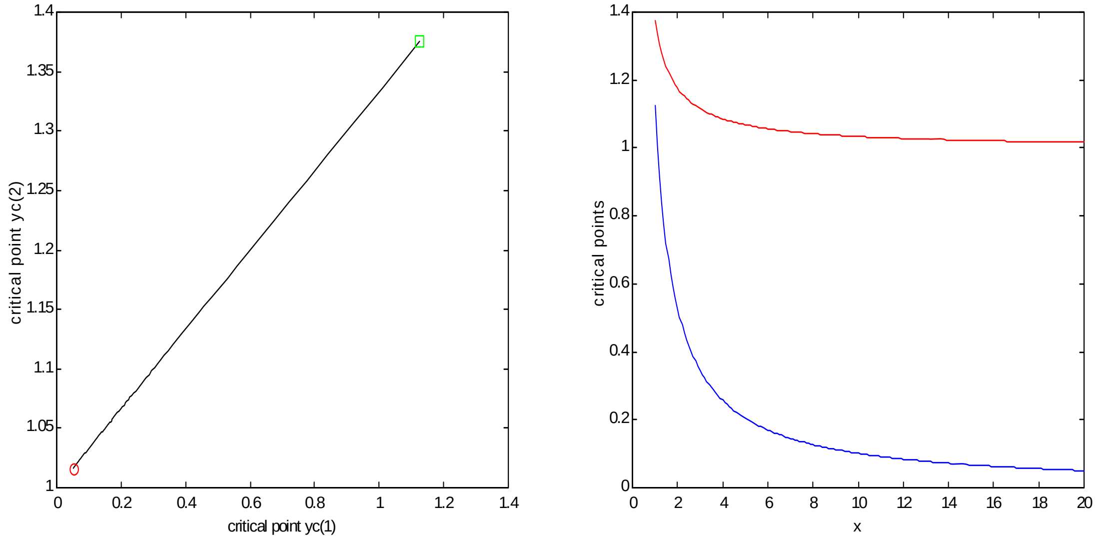
The eigenvalues and eigenvectors are plotted below as a function of x.



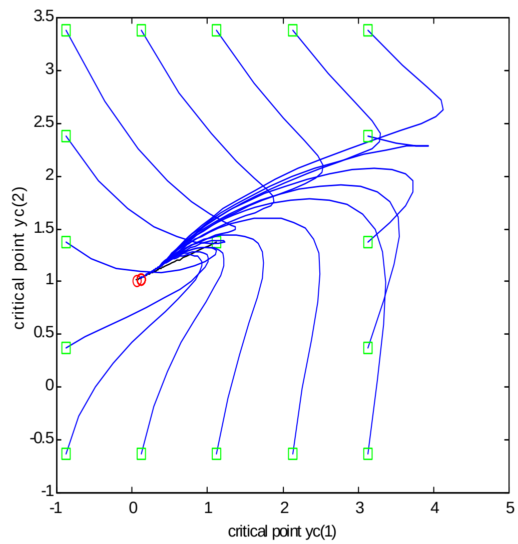


We can see that the eigenvalues start small and negative; one linearly decreases and the other appears to exponentially decrease. The eigenvectors appear to be approaching asymptotic values of [1,0] and [0,1] respectively. One important observation is that the eigenvalues are always negative and always purely real. Therefore the system will always exhibit the behavior of a stable improper node.

The critical points are also a function of x and are obtained from eqn (IV.3). Plots of the critical path are given below, both as a function of *x* and parametrically.



The critical points are following exponential decays to a given value. At long times, this critical point will no longer move and will result in a stationary state.



Phase Plots: Example IV.2. Stable improper node following moving steady state.

This figure shows that we have a stable node for a critical point. We would expect this because our eigenvalues are real and negative. The only difference between this case and the case where the matrix $\underline{\underline{A}}$ is constant is that now our critical point is mobile. The trajectories follow along behind it.

V. Non-Linear ODEs

Most of the world's problems are non-linear. What good does our previous analysis of linear ODEs do for us? Well, it tells us the five types of critical points. It tells us that the critical points are the steady-state solutions. It tells us that we can understand the behavior of a system of ODEs by looking at a phase plot.

We write the generic system of nonlinear ODEs as

$$\frac{d\underline{y}}{dx} = \underline{f}(x, \underline{y}) \tag{V.1}$$

For this system, we can still determine critical points and we can determine the stability of the critical points. We can also determine the type of critical point, at least in the local vicinity around each critical point. Therefore, in the nonlinear case, we cannot determine the type and stability of the critical point(s) until we first find the critical point(s).

We use the same generalized definition of the critical points,

$$\frac{dy_i}{dx} = 0 \text{ for all } 1 \le i \le n \tag{III.2}$$

For the nonlinear system, this results in a system of non-linear algebraic equations

$$\underline{f}(x, \underline{y}) = 0 \tag{V.2}$$

which we would solve numerically using the Newton-Raphson method or some other appropriate numerical method. The solution to this equation is $\underline{f}(x, \underline{y}_c) = 0$ where $\underline{y}_c$ is a critical point.

We next expand the all functions in a multivariate Taylor series about the critical point and truncate after the linear terms. For a system of 2 ODEs, this is explicitly

$$\begin{aligned} f_1(y_1(x), y_2(x)) &= f_1(y_{1c}, y_{2c}) + \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c}(y_1(x) - y_{1c}) + \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c}(y_2(x) - y_{2c}) \\ f_2(y_1(x), y_2(x)) &= f_2(y_{1c}, y_{2c}) + \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c}(y_1(x) - y_{1c}) + \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c}(y_2(x) - y_{2c}) \end{aligned} \tag{V.3}$$

We have effectively linearized the functions and the linearized ODEs are now,

$$\frac{dy_1}{dx} = f_1(y_{1c}, y_{2c}) + \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c}(y_1(x) - y_{1c}) + \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c}(y_2(x) - y_{2c})$$

$$\frac{dy_2}{dx} = f_2(y_{1c}, y_{2c}) + \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} (y_1(x) - y_{1c}) + \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} (y_2(x) - y_{2c}) \tag{V.4}$$

We can write this in matrix notation as

$$\frac{d\underline{y}}{dx} = \underline{\underline{J}}\,\underline{y} + \underline{b} \tag{V.5}$$

where $\underline{\underline{J}}$ is the Jacobian, the matrix of first partial derivatives, which has the definition for the 2x2 as

$$\underline{\underline{J}} \equiv \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix} \tag{V.6}$$

and $\underline{b}$ is a vector of constants,

$$\underline{b} = \begin{bmatrix} f_1(y_{1c}, y_{2c}) - \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} y_{1c} - \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} y_{2c} \\ f_2(y_{1c}, y_{2c}) - \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} y_{1c} - \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} y_{2c} \end{bmatrix} \tag{V.7}$$

The Jacobian used here is the same Jacobian that is used in the Newton-Raphson method. Once we have linearized the ODE, we can use the straightforward procedure described in Section II (or III if we have more than 2 equations) to determine the eigenvalues and eigenvectors of the Jacobian.

$$\det\left(\underline{\underline{J}} - \lambda \underline{\underline{I}}\right) = 0 \tag{V.8}$$

From here, we have the eigenvalues and eigenvectors of the problem around a given critical point. This procedure can be repeated for each critical point in the system.

Example V.1. Nonlinear ODEs – Saddle Point

Consider the set of ODEs

$$\frac{dy_1}{dx} = y_1(x)^2 - 3y_2(x) - 4$$
$$\frac{dy_2}{dx} = -3y_1(x) + y_2(x)$$

A critical point of this system is

$$\underline{y}_c = \begin{bmatrix} -0.4244 \\ -1.2733 \end{bmatrix}$$

The Jacobian of this system is

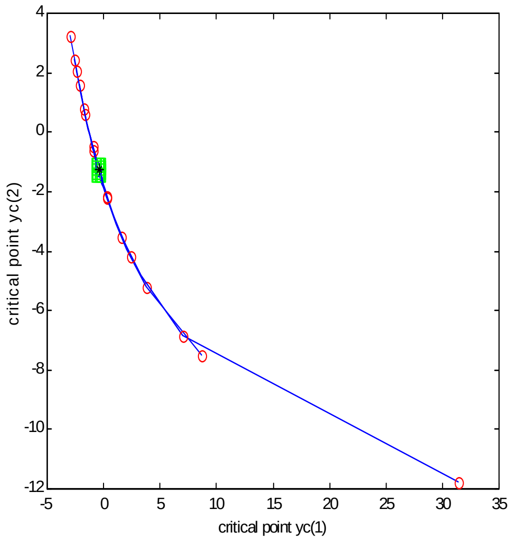
$$\underline{\underline{J}} = \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} 2y_{1c} & -3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -0.8488 & -3 \\ -3 & 1 \end{bmatrix}$$

The eigenvalues and eigenvectors of the Jacobian matrix are

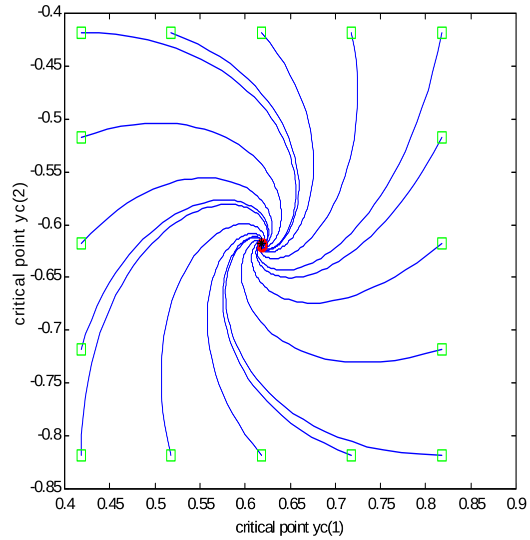
$$\underline{\underline{\Lambda}} = \begin{bmatrix} -3.0636 & 0 \\ 0 & 3.2148 \end{bmatrix} \qquad \underline{\underline{W}} = \begin{bmatrix} -0.8045 & -0.5939 \\ -0.5939 & 0.8045 \end{bmatrix}$$

The eigenvalues are real, therefore we either have a node or a saddle point. The eigenvalues are not all of the same sign. Therefore, we have a saddle point. All saddle points are unstable.

The phase plot is shown below. The starting points (green squares) all lead away from the critical point (black star) to their respective ending points at x=1.0 (red circles). There is only one path away from the critical point, since the system is a saddle and the other eigenvector led to the critical point. The eigenvector leading away from the critical point is not a straight line since the problem is nonlinear.



Phase Plot: Example V.1. (Unstable) Saddle Point.



Phase Plot: Example V.2. Stable Spiral Point.

Example V.2. Nonlinear ODEs – Stable Spiral Point

Consider the set of ODEs

$$\frac{dy_1}{dx} = -y_1(x)^2 + y_2(x) + 1$$
$$\frac{dy_2}{dx} = -y_1(x) - y_2(x)$$

A critical point of this system of equations is $\underline{y}_c = \begin{bmatrix} 0.6180 \\ -0.6180 \end{bmatrix}$

In order to determine the eigenvalues, we again linearize the system of ODEs with a Taylor series expansion. The Jacobian of the linearized problem is

$$\underline{\underline{J}} \equiv \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} -2y_{1c} & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1.2360 & 1 \\ -1 & -1 \end{bmatrix}$$

The eigenvalues and eigenvectors of the Jacobian matrix are

$$\underline{\underline{\Lambda}} = \begin{bmatrix} -1.1180 + 0.9930i & 0 \\ 0 & -1.1180 - 0.9930i \end{bmatrix} \qquad \underline{\underline{W}} = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.0834 + 0.7022i & 0.0834 - 0.7022i \end{bmatrix}$$

The eigenvalues are complex conjugates , therefore we have a spiral point. The real component of the eigenvalues are less than zero. Therefore, the point is stable. We should observe the behavior of a stable spiral point, at least near the critical point. Plots of trajectories starting from initial conditions near the critical point yield confirm this prediction.

Example V.3. Nonlinear ODEs – Multiple Steady States

Consider a continuously stirred tank reactor (CSTR) with a mass balance on reactant *A* given by

$$V\frac{dC_A}{dt} = QC_{A,in} - QC_A - VkC_A$$

where these terms represent from left to right accumulation of A in the reactor, flow of A into the reactor, flow of A out of the reactor and consumption of A via a first-order chemical reaction. The reaction rate constant, k, is a function of temperature,

$$k = k_o \exp\left(\frac{-E_a}{RT}\right)$$

Coupled to this mass balance is an energy balance,

$$V\rho C_p \frac{dT}{dt} = Q\rho C_p T_{in} - Q\rho C_p C_A - \Delta H_R VkC_A$$

which has the same accumulation, input, output and generation terms in it. For the sake of convenience, we choose to rewrite the mass balance in terms of a dimensionless variable, the extent of reaction, which is bounded between 0 (no reaction) and 1 (complete reaction) and is defined as

$$X \equiv 1 - \frac{C_A}{C_{A,in}}$$

so that the first ODE can be rewritten as

$$\frac{dX}{dt} = -\frac{1}{C_{A,in}}\frac{dC_A}{dt}$$

The variables in these two nonlinear ODEs are the extent of reaction, X , and temperature, *T*. Everything else is assumed constant in this problem. In this problem, the reaction is exothermic (it generates heat) or in other words $\Delta H_R < 0$, so that the generation term in the energy balance is positive. This is a key element of this system.

A sample input file is provided below.

```
function dydt = funkeval(y)
x = y(1);                    % extent of reaction
T = y(2);                    % Temperature K
Cin = 3.0;                   % inlet concentration mol/l
C = Cin*(1-x);               % concentration
Q = 60e-3;                   % volumetric flowrate l/s
R = 8.314;                   % gas constant J/mol/K
Ea = 62800;                  % activation energy J/mol
ko = 4.48e+6;                % reaction rate prefactor 1/s
k = ko*exp(-Ea/(R*T));       % reaction rate constant 1/s
V = 18;                      % reactor volume l
Cp = 4.19e3;                 % heat capacity J/kg/K
 Tin = 298;             % inlet feed temperature K
 Tref = 298;            % thermodynamic reference temperature K
 DHr = -2.09e5;         % heat of rxn J/mol
 rho = 1.0;             % density kg/l
 dydt(1) = 1/V*(Q*Cin - Q*C - k*C*V);  % mass balance mol/s
 dydt(2) = 1/(Cp*rho*V)*(Q*Cp*rho*Tin  - Q*Cp*rho*T -
          DHr*k*C*V);        % NRG balance J/s
 dydt(1) = -1/Cin*dydt(1);   % convert from concentration to
                             extent
```

These are the design equations for a continuously stirred-tank reactor with a single first-order exothermic reaction, operating under adiabatic conditions.

The unknowns are the extent of reaction and the temperature.

This problem has three steady states. The critical points of this system of equations are

$$\underline{y}_c = \begin{bmatrix} x \\ T \end{bmatrix},\ \underline{y}_{c,1} = \begin{bmatrix} 0.0159 \\ 300.4 \end{bmatrix},\ \underline{y}_{c,2} = \begin{bmatrix} 0.3335 \\ 347.9 \end{bmatrix},\ \text{and}\ \underline{y}_{c,3} = \begin{bmatrix} 0.9828 \\ 445.1 \end{bmatrix}$$

In order to determine the eigenvalues, we again linearize the system of ODEs with a Taylor series expansion. In this case, since we have more than one critical point, we must evaluate the Jacobian at each of the steady states. The Jacobian of the linearized problem is

$$\underline{\underline{J}} \equiv \begin{bmatrix} \left.\frac{df_1}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_1}{dy_2}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dy_1}\right|_{\underline{y}_c} & \left.\frac{df_2}{dy_2}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} \left.\frac{df_1}{dx}\right|_{\underline{y}_c} & \left.\frac{df_1}{dT}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dx}\right|_{\underline{y}_c} & \left.\frac{df_2}{dT}\right|_{\underline{y}_c} \end{bmatrix} = \begin{bmatrix} \left.\frac{df_1}{dC}\right|_{\underline{y}_c} \frac{dC}{dx} & \left.\frac{df_1}{dT}\right|_{\underline{y}_c} \\ \left.\frac{df_2}{dC}\right|_{\underline{y}_c} \frac{dC}{dx} & \left.\frac{df_2}{dT}\right|_{\underline{y}_c} \end{bmatrix}$$

$$\underline{\underline{J}} = \begin{bmatrix} -\frac{Q}{V} - k & \frac{kE_a}{C_{in}RT^2} \\ \frac{\Delta H_r C_{in} k}{C_p \rho} & -\frac{Q}{V} - \frac{\Delta H_r C k E_a}{C_p \rho R T^2} \end{bmatrix}$$

We evaluate this Jacobian at each critical point and get the eigenvalues. For the first critical point, we have

$$\underline{\underline{J}} = \begin{bmatrix} -0.0034 & 0.0000 \\ -0.0081 & -0.0027 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -0.0034 & 0 \\ 0 & -0.0027 \end{bmatrix} \qquad \underline{\underline{W}} = \begin{bmatrix} -0.0866 & -0.0021 \\ -0.9962 & -1.0000 \end{bmatrix}$$

The eigenvalues of the first critical point are real and negative. Therefore, the first critical point will behave like a stable improper node.

For the second critical point, we have

$$\underline{\underline{J}} = \begin{bmatrix} -0.0050 & 0.0000 \\ -0.2495 & 0.0070 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -0.0042 & 0 \\ 0 & 0.0063 \end{bmatrix} \qquad \underline{\underline{W}} = \begin{bmatrix} -0.0452 & -0.0031 \\ -0.9990 & -1.0000 \end{bmatrix}$$

The eigenvalues of the second critical point are real. One is positive and one is negative. Therefore, the second critical point will behave like a saddle point, which is always unstable.

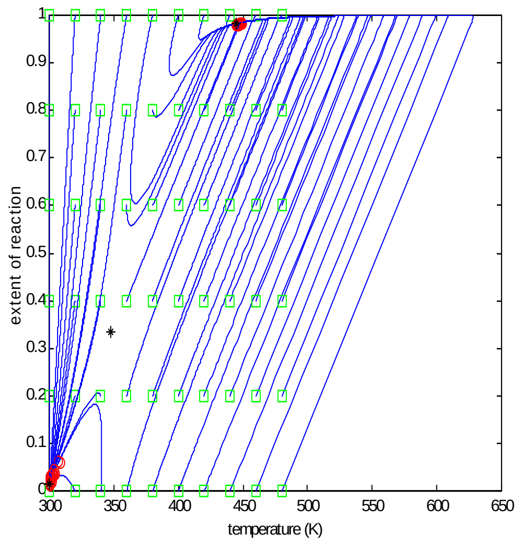
For the third and final critical point, we have

$$\underline{\underline{J}} = \begin{bmatrix} -0.1944 & 0.0024 \\ -28.5878 & 0.0154 \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} -0.0895 + 0.2417\mathrm{i} & 0 \\ 0 & -0.0895 - 0.2417\mathrm{i} \end{bmatrix}$$

$$\underline{\underline{W}} = \begin{bmatrix} 0.0037 - 0.0085\mathrm{i} & 0.0037 + 0.0085\mathrm{i} \\ 1.0000 & 1.0000 \end{bmatrix}$$

The eigenvalues of the third critical point are complex. The real components of the eigenvalues are negative. Therefore, the third critical point will behave like a stable spiral point.

Some trajectories, based on different initial conditions (different initial concentrations and reactor temperatures) are shown below. The trajectories start at green squares and end at red circles. The time that transpired along each trajectory is 1 minute.



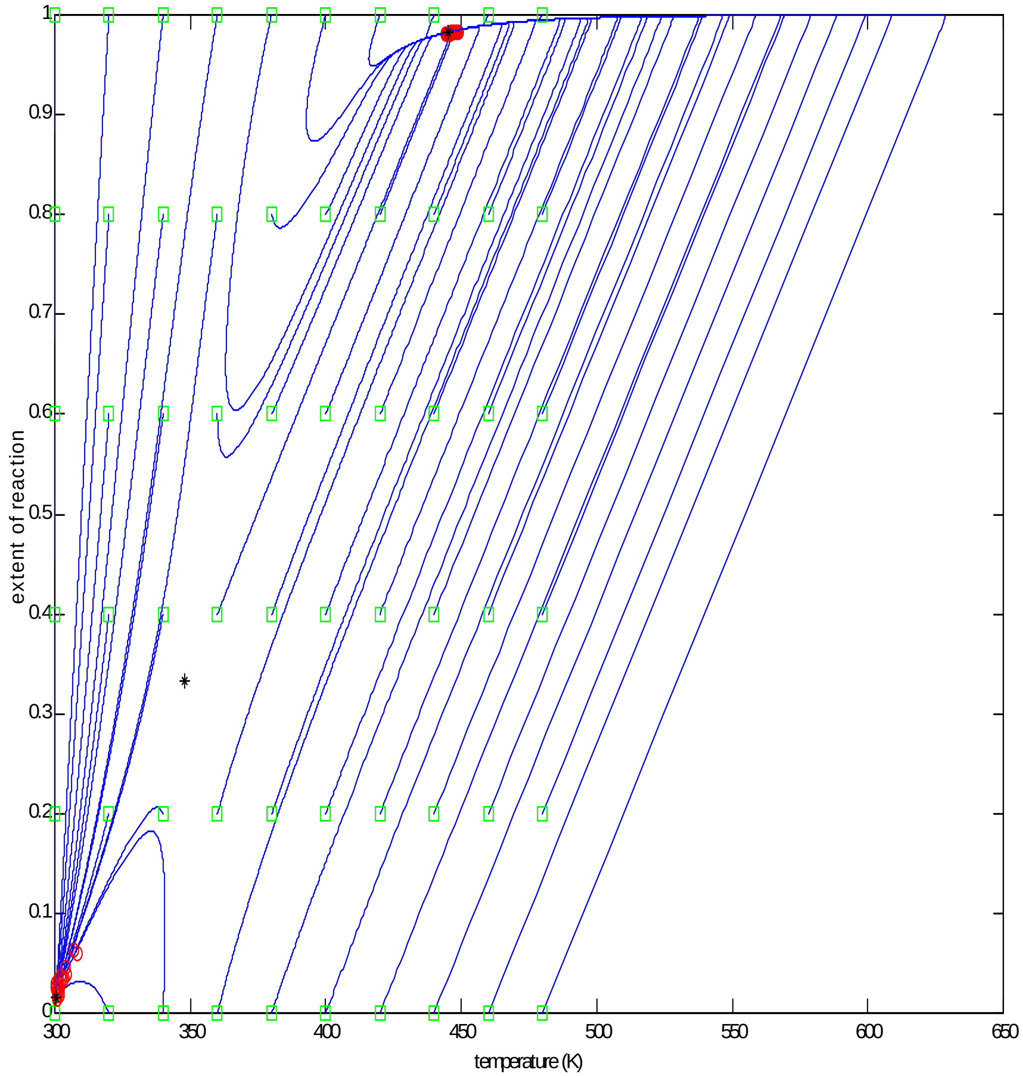
(a) initial temperatures = 300 < T < 500
initial extent of reactions 0 < x < 1
duration of trajectory = 1200 sec
(larger version of plot available on last page of this section)

From the trajectory plots given above we can determine the nature of the critical points (steady state solutions in this example). The low-conversion/low-temperature and the high-conversion/high-temperature solutions are stable attractors. The intermediate solution is an unstable node. The eigenvalues associated with the attractors are less than zero. At least one eigenvalue associated with the unstable node is negative.

We can also see some qualitative information about the system. We can define roughly the basins of attraction for the two attractors. For the coarse grid we used, any initial temperature of 380 K or higher converged to the high critical point. Any initial temperature of 340 K or lower converged to the low critical point. For initial temperatures of 360K, those with high initial extents of reaction proceeded to the low root; those with low initial extents of reaction converged to the high root.

The trajectories that led to the low root, approached with a final tangents that appeared to be nearly parallel to the difference vector between the low and middle root. The trajectories that led to the high root, approached with two different final tangents. The first seemed to be nearly parallel to the difference vector between the high and middle root. The second, which most of the trajectories followed, appeared to be roughly perpendicular to the first.

Some initial conditions with low initial extent of reaction and low temperature, proceeded through temperatures higher than the high root on their way to the high root. This is because the reactor is full of unreacted product. It reacts initially, which, since the reaction is exothermic, heats up the reactor. It then takes some time for new feed to enter and cool the reactor to its steady state temperature.



extent of reaction
temperature (K)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
300
350
400
450
500
550
600
650