Analytical Integration

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Disclaimer:
This course is a targeted at developing practical problem solving skills in scientists and engineers. In this brief lecture module, we put aside any pretense of mathematical pride and identify the most practical method for the analytical evaluation of integrals.

Problem Statement:
Our problem of interest is the evaluation of the following integral, *I*, in which we have identified the integrand, $f(x)$, the dummy variable of integration, *x*, the lower and upper limits of integration, *a* and *b* respectively, and the indefinite integral, $F(x)$,

$$I = \int_a^b f(x)dx = F(x=b) - F(x=a) \quad (1)$$

In my experience, it is always superior to have an analytical form of the integral than a numerical evaluation. An analytical form is advantageous for three reasons. First, there may be insight into the behavior of the physical system being modeled in the functional form of the integral. Second, the analytical method is exact and does not contain approximations introduced by numerical techniques. Third, it is likely computationally more efficient to evaluate the integral analytically than it is to do the same numerically. The analytical result of a definite integral requires two function evaluations of the indefinite integral, $F(x)$. The numerical result requires many function evaluations of the integrand, $f(x)$.

Analytical Techniques:
Three routes to analytical solutions are discussed below.

1. Know your Integral Calculus
Integrals can be analytically evaluated based on the knowledge of the mathematical rules of integration. For simple functional forms of the integrand this is the quickest route. For more complicated integrands where (i) the existence of an analytical functional form of the indefinite integral, $F(x)$, is in doubt and/or (ii) the functional form of $f(x)$ is beyond our knowledge of integral calculus, we need another route.

2. Tables of Integrals
In the stone age, cave men identified analytical forms of indefinite integrals through very useful books known as "Tables of Indefinite Integrals". These tomes were stored in libraries. Inside

them, one could find lists of indefinite integrals. Presumably some of these tomes are still available in libraries and can still be accessed for this purpose.

3. Symbolic Integrators

In the computer age, the collaboration between mathematicians with knowledge of the rules of integral calculus and computer scientists has resulted in software capable of providing the analytical forms of integrals. These codes are called symbolic manipulators because they manipulate algebraic variables rather than numeric values.

One freely accessible software for symbolic integration the Wolfram Online Integrator available at http://integrals.wolfram.com/index.jsp. This integrator has been freely available via the internet for more than a decade.

To my knowledge, this symbolic manipulator provides one of three outputs.

a. An analytical form exists and is output.

Example:

$$\int \frac{\sec(x)\tan(x)}{9+4\sec^2(x)}dx = -\frac{1}{6}\tan^{-1}\left(\frac{3\cos(x)}{2}\right)$$

b. No analytical form exists but a common function exists.

$$\int \exp\left(-x^2\right)dx = \frac{1}{2}\sqrt{\pi}\,erf(x)$$

where the error function is a common mathematical integral with no analytical solution,

$$erf(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt$$

Most languages have an internal command for the error function, just as they do for *sin* or *log*.

c. No known analytical form exists.

$$I = \int \exp\left(-\sin\left(x^2\right)\right)dx$$

Inputting this integrand results in the following message, "Mathematica could not find a formula for your integral. Most likely this means that no formula exists."