Midterm Examination Solutions
February 26, 2019

1. Dynamic Behavior and Stability of a Pendulum with Drag

Consider a simple model of a rigid pendulum moving in an atmosphere with non-negligible drag. The model describing the motion of the pendulum is given by

$$\frac{d^2\theta}{dt^2} = \frac{1}{m\,\ell}\left(-mgsin(\theta) - \frac{1}{2}\rho\ell\frac{d\theta}{dt}C_D A\right)$$

where θ is the angle in radians defined as the deviation from normal, *m* is the mass of the pendulum, ℓ is the length of the pendulum, *g* is acceleration due to gravity, ρ is the density of the medium in which the pendulum swings, C_D is the drag coefficient, *A* is the cross-sectional area of the pendulum, and *t* is time.

Consider the following numerical parameters, $m = 1.0$ kg, $\ell = 1.0$ m, $g = 9.8\, m/s^2$, $\rho_{vacuum} = 0.0\, kg/m^3$, $\rho_{air} = 1.225\, kg/m^3$, $\rho_{water} = 1000.0\, kg/m^3$, $A = 0.01\, m^2$ and $C_D = 0.47$.

For parts (a) through (f) of the problem, consider the following initial conditions, at time *t* = 0,
$\theta = \frac{\pi}{2}$ and $\frac{d\theta}{dt} = 0$.

(a) Is this ODE linear or nonlinear?
(b) Convert the second order ODE to a system of first order ODEs.
(c) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in vacuum. Sketch the behavior.
(d) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in air. Sketch the behavior.
(e) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in water. Sketch the behavior.
(f) Determine the critical point of the system.
(g) Construct the Jacobian of the system of ODEs and evaluate it at the critical point.
(h) Report the eigenvalues of the Jacobian at the critical point for vacuum, air and water.
(i) State the stability of the systems based on the solution of the ODEs and the eigenvalues. Sketch phase plot, if necessary.
(j) extra credit: Is it possible to change the stability of the pendulum? Is it possible to lose all oscillatory behavior in the pendulum?

Solution:

(a) Is this ODE linear or nonlinear?

This ODE is nonlinear in θ because of the sine function.

(b) Convert second order ODE to a system of first order ODEs.

There is a three-step process to accomplish this transformation. First, identify each new variable.

$$y_1 = \theta$$
$$\boldsymbol{y_2} = \frac{d\theta}{dt}$$

Second, write the ODE for each new variable.

$$\frac{dy_1}{dt} = y_2$$
$$\frac{dy_2}{dt} = \frac{1}{m\,\ell}\left(-mgsin(y_1) - \frac{1}{2}\rho \ell y_2 C_D A\right)$$

Third, write the ICs for each variable.

$$\boldsymbol{y_1(t_o)} = \frac{\pi}{2}$$
$$\boldsymbol{y_2(t_o)} = 0$$

(c) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in vacuum. Sketch the behavior.

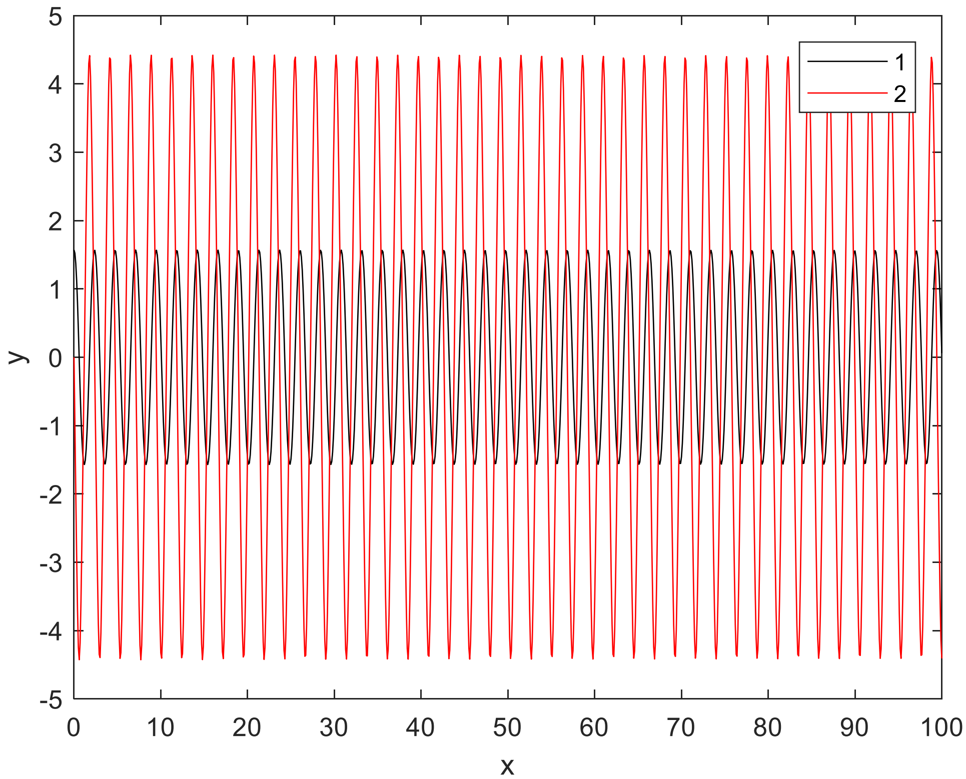
I will use the Classical fourth-order Runge-Kutta method to solve this problem.
I modified the input function for rk4n.m

```
function dydx = funkeval(x,y);
m = 1.0; %kg
len = 1.0; %m
A = 0.01; % m^2
pi = 2.0*asin(1.0);
g = 9.8; % m/s^2
rho_vacuum = 0.0; % kg/m^3 (vacuum)
rho_air = 1.225; % kg/m^3 (air)
rho_water = 1000.0; % kg/m^3 (water)
rho = rho_vacuum;
CD = 0.47;
dydx(1) = y(2);
dydx(2) = 1.0/(m*len)*(-m*g*sin(y(1)) - 0.5*rho*len*y(2)*CD*A);
```

At the command line prompt, I typed

```
>> [x,y]=rk4n(1000,0,100,[asin(1.0),0.0]);
```

This command yielded the following plot.

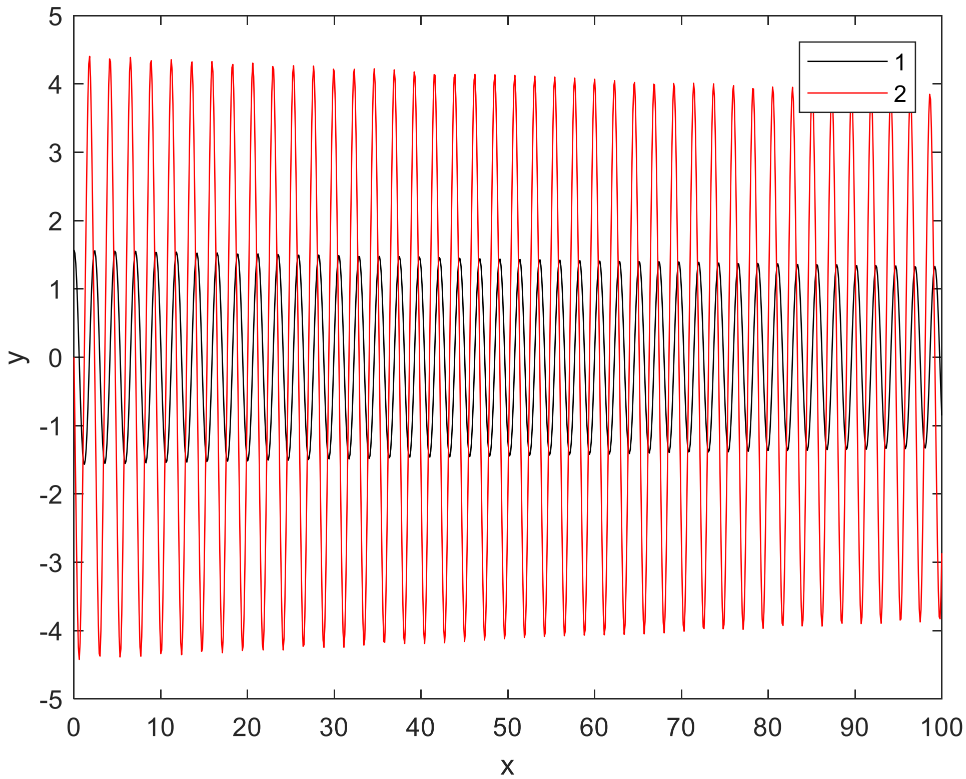


(d) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in air. Sketch the behavior.

Using the same code as in part (c), I changed the density to that for air. I typed the same command.

```
>> [x,y]=rk4n(1000,0,100,[asin(1.0),0.0]);
```

This command yielded the following plot.

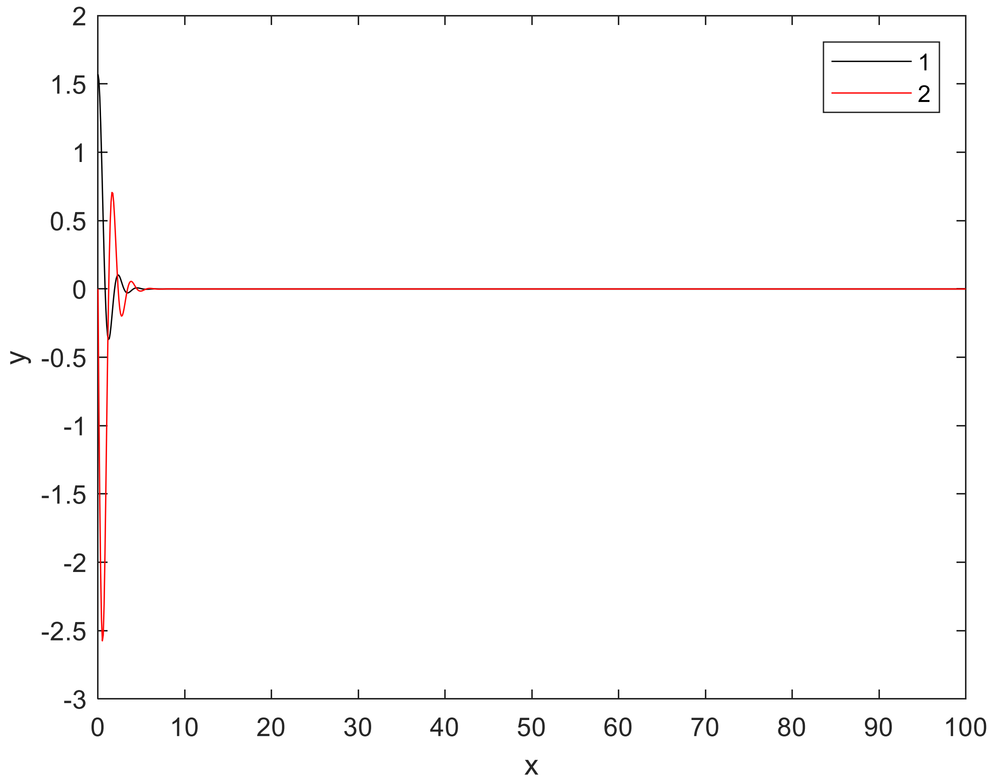


(e) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in water. Sketch the behavior.

Using the same code as in part (c), I changed the density to that for water. I typed the same command.

```
>> [x,y]=rk4n(1000,0,100,[asin(1.0),0.0]);
```

This command yielded the following plot.



(f) Determine the critical point of the system.

To determine the critical point, set the ODEs to zero and solve the resulting system of non-linear algebraic equations.

$$\frac{dy_1}{dt} = 0 = y_2$$
$$\frac{dy_2}{dt} = 0 = \frac{1}{m\,\ell}\left(-mgsin(y_1) - \frac{1}{2}\rho\ell y_2 C_D A\right)$$

By inspection the first equation yields $y_2 = 0$. The second equation then becomes

$$0 = sin(y_1)$$

which has a root at $y_1 = 0$.

Thus the critical point is at $(y_1, y_2) = (0,0) = \left(\theta, \frac{d\theta}{dt}\right)$

(g) Construct the Jacobian of the system of ODEs and evaluate it at the critical point.

$$\boldsymbol{J} = \begin{bmatrix} \frac{df_1}{dy_1} & \frac{df_1}{dy_2} \\ \frac{df_2}{dy_1} & \frac{df_2}{dy_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}cos(y_1) & -\frac{1}{2}\frac{\rho C_D A}{m} \end{bmatrix}$$

If we evaluate the Jacobian at the critical point $(y_1, y_2) = (0,0)$ we have

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{1}{2}\frac{\rho C_D A}{m} \end{bmatrix}$$

(h) Report the eigenvalues of the Jacobian at the critical point.

$$\boldsymbol{J} - \lambda\boldsymbol{I} = \begin{bmatrix} -\lambda & 1 \\ -\frac{g}{\ell} & -\frac{1}{2}\frac{\rho C_D A}{m} - \lambda \end{bmatrix}$$

The determinant of this matrix is the characteristic equation.

$$\det(\boldsymbol{J} - \lambda\boldsymbol{I}) = 0 = -\lambda\left(-\frac{1}{2}\frac{\rho C_D A}{m} - \lambda\right) + \frac{g}{\ell} = \lambda^2 + \frac{1}{2}\frac{\rho C_D A}{m}\lambda + \frac{g}{\ell}$$

Using the quadratic formula to find the roots of a quadratic polynomial, we have

$$\lambda = \frac{-\frac{1}{2}\frac{\rho C_D A}{m} \pm \sqrt{\left(\frac{1}{2}\frac{\rho C_D A}{m}\right)^2 - 4\frac{g}{\ell}}}{2}$$

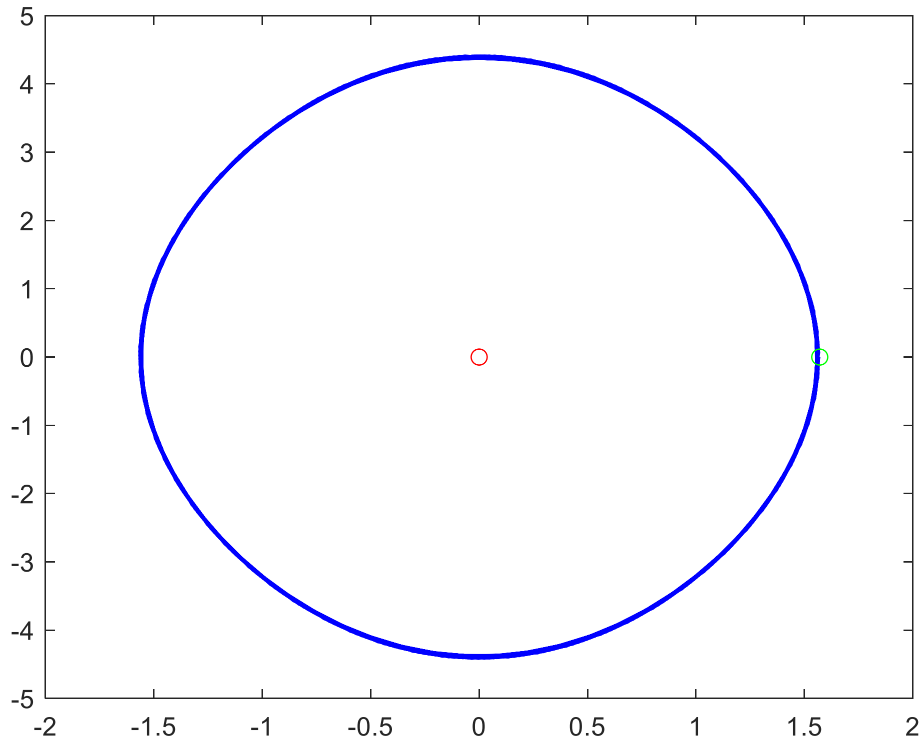
When the density is zero, as is the case in vacuum, the eigenvalues are

$$\lambda = \pm\sqrt{\frac{g}{\ell}}\, i$$

For vacuum, $\lambda = \pm 3.1305i$.
For air, $\lambda = -0.0014 \pm 3.1305i$.
For water, $\lambda = -1.1750 \pm 2.9016i$.

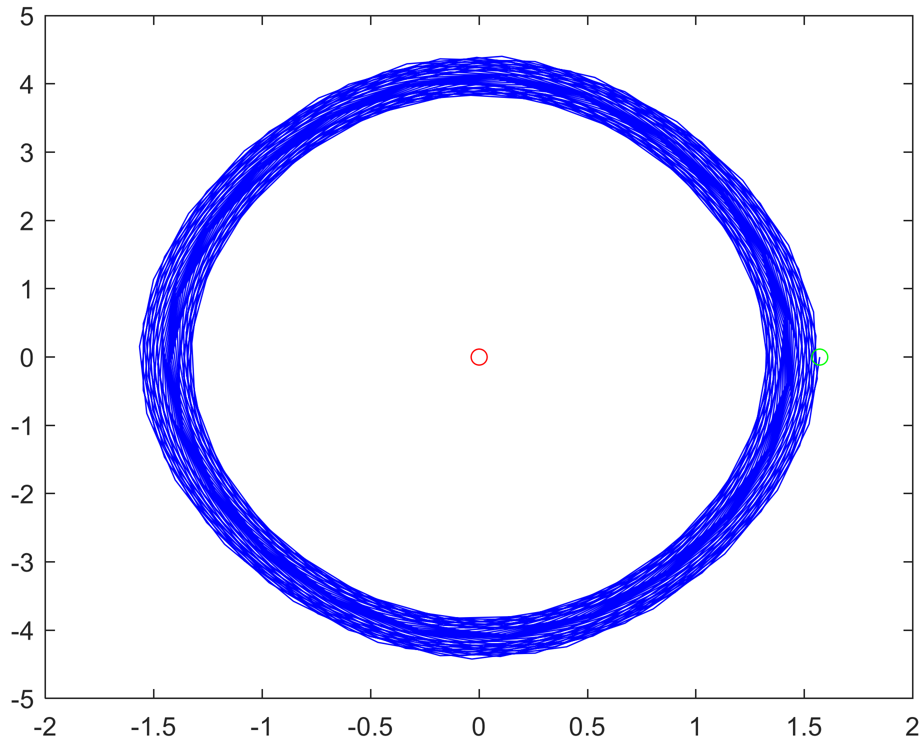
(i) State the stability of the systems based on the solution of the ODEs and the eigenvalues. Sketch phase plot, if necessary.

In vacuum, the pendulum is a center. The eigenvalues are purely imaginary. It will oscillate forever. A phase plot is shown below.

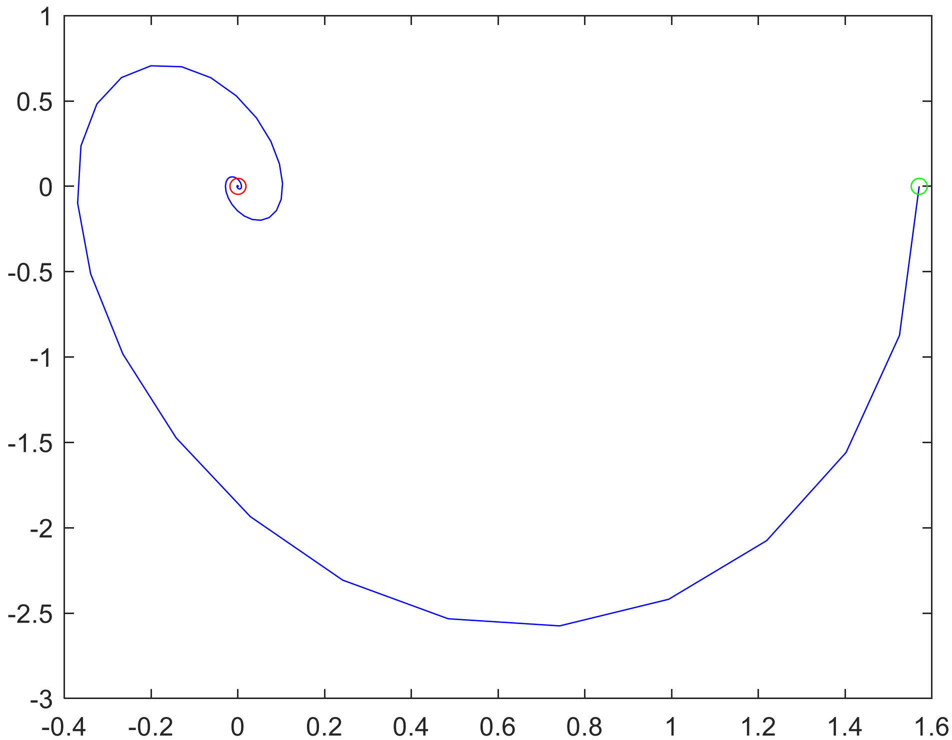


In air and water, the pendulum is a stable, spiral. The eigenvalues are complex and the real components are negative.

A phase plot of the air system is shown below. It has a slow decay.



A phase plot of the water system is shown below. It has a relatively fast decay.



(j) extra credit: Is it possible to change the stability of the pendulum? Is it possible to have a non-zero density without decaying to the critical point?

Is it possible to change the stability of the pendulum?

No, the real component of the eigenvalues is $real(\lambda) = -\frac{1}{4}\frac{\rho C_D A}{m}$. This is always negative since density, mass, area and the drag coefficient are physical properties that cannot take on negative values.

Is it possible to lose all oscillatory behavior in the pendulum?

We will lose any imaginary component of the trajectory when the discriminant, $\left(\frac{1}{2}\frac{\rho C_D A}{m}\right)^2 - 4\frac{g}{\ell}$, is set to zero.

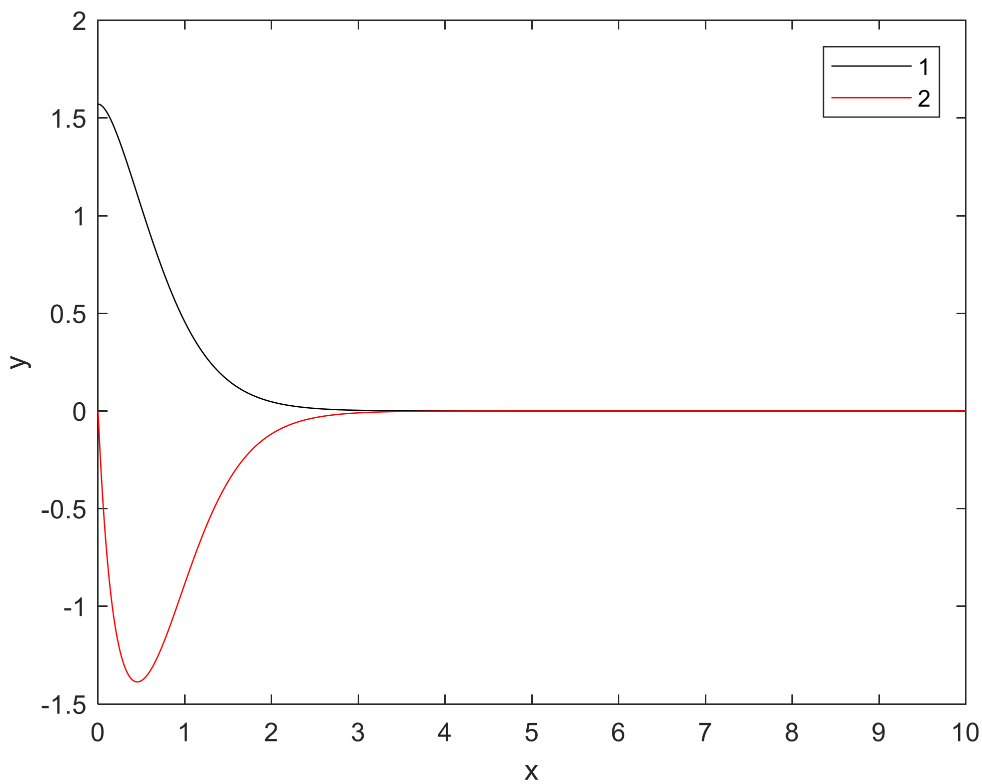
$$\left(\frac{1}{2}\frac{\rho C_D A}{m}\right)^2 - 4\frac{g}{\ell} = 0$$

Solving for density, we have

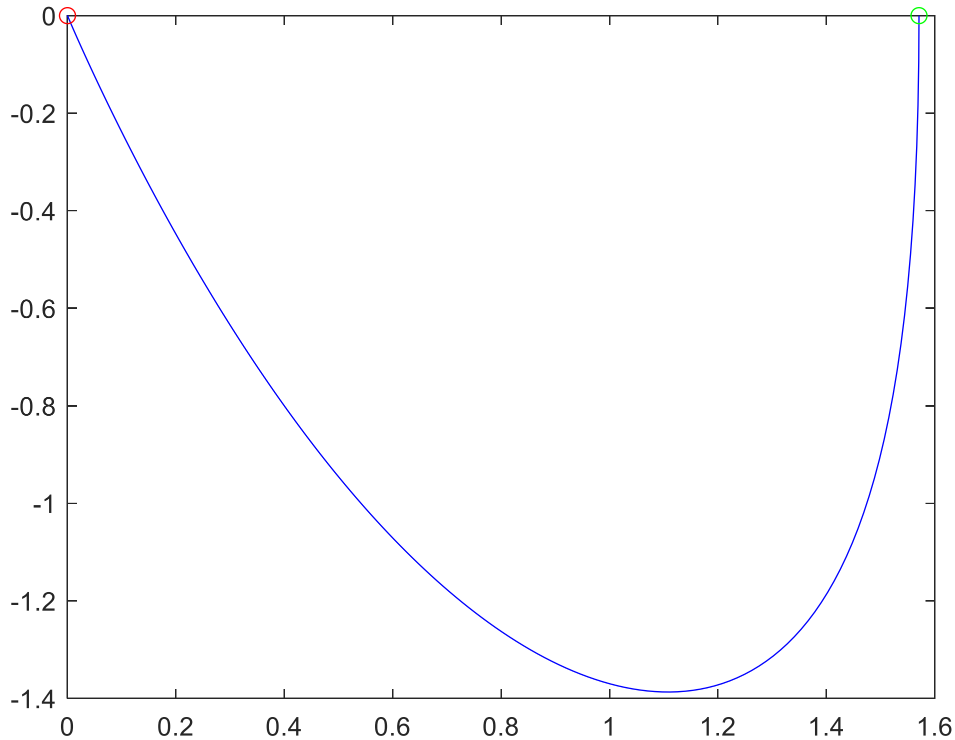
$$\rho = \frac{m}{C_D A} 4\sqrt{\frac{g}{\ell}}$$

If we set the density to a value higher than this limit, we should have purely real eigenvalues and we should see the behavior of a stable, improper node.

In the example below, I set the density to 1% higher than this limit, $\rho_{dense} = 2690.9\ kg/m^3$, which yields purely real eigenvalues of -2.7180 and -3.6056 and the following plot



A phase plot is shown below. It has no oscillatory behavior.


0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6