Midterm Examination
February 26, 2019

1. Dynamic Behavior and Stability of a Pendulum with Drag

Consider a simple model of a rigid pendulum moving in an atmosphere with non-negligible drag. The model describing the motion of the pendulum is given by

$$\frac{d^2\theta}{dt^2} = \frac{1}{m\,\ell}\left(-mgsin(\theta) - \frac{1}{2}\rho\ell\frac{d\theta}{dt}C_D A\right)$$

where θ is the angle in radians defined as the deviation from normal, *m* is the mass of the pendulum, ℓ is the length of the pendulum, *g* is acceleration due to gravity, ρ is the density of the medium in which the pendulum swings, C_D is the drag coefficient, *A* is the cross-sectional area of the pendulum, and *t* is time.

Consider the following numerical parameters, $m = 1.0$ kg, $\ell = 1.0$ m, $g = 9.8\ m/s^2$, $\rho_{vacuum} = 0.0\ kg/m^3$, $\rho_{air} = 1.225\ kg/m^3$, $\rho_{water} = 1000.0\ kg/m^3$, $A = 0.01\ m^2$ and $C_D = 0.47$.

For parts (a) through (f) of the problem, consider the following initial conditions, at time *t* = 0, $\theta = \frac{\pi}{2}$ and $\frac{d\theta}{dt} = 0$.

(a) Is this ODE linear or nonlinear?
(b) Convert the second order ODE to a system of first order ODEs.
(c) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in vacuum. Sketch the behavior.
(d) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in air. Sketch the behavior.
(e) Numerically solve for the dynamic behavior of the pendulum for 100 seconds for the pendulum operating in water. Sketch the behavior.
(f) Determine the critical point of the system.
(g) Construct the Jacobian of the system of ODEs and evaluate it at the critical point.
(h) Report the eigenvalues of the Jacobian at the critical point for vacuum, air and water.
(i) State the stability of the systems based on the solution of the ODEs and the eigenvalues. Sketch phase plot, if necessary.
(j) extra credit: Is it possible to change the stability of the pendulum? Is it possible to lose all oscillatory behavior in the pendulum?