Midterm Examination Solutions
February 26, 2015

1. Dynamic Behavior and Stability Analysis of the Moon

Consider a simple, two dimensional model of the moon's orbit, which focuses exclusively on the effect of the Earth on the moon and neglects the effect of the moon on the Earth. In this model, the Earth is stationary at a position of $(x_E, y_E) = (0,0)$. This motion is described by two second-order non-linear differential equations,

$$a_x = \frac{d^2 x_M}{dt^2} = \frac{F_x}{m_M} = -g\frac{m_M m_E}{m_M}\frac{1}{r^3}x_M = -\frac{gm_E}{r^3}x_M$$

$$a_y = \frac{d^2 y_M}{dt^2} = \frac{F_y}{m_M} = -g\frac{m_M m_E}{m_M}\frac{1}{r^3}y_M = -\frac{gm_E}{r^3}y_M$$

where m_M and m_E are the mass of the moon and the earth respectively, x_M and y_M are the coordinates of the position of the moon with respect to Earth, g is the gravitational constant, and r is the radial separation between the earth and moon, $r = \sqrt{x_M^2 + y_M^2}$.

We shall work this problem in reduced units, where time is measure in years, length is measured in units of the distance between the moon and Earth, e.g. (r *should* be 1), $m_M = 1$, $m_E = 81.28$ moon units and g = 85.8596.

For parts (a) through (f) of the problem, consider the following initial conditions, at time t = 0,

$$x_M = 1,\ y_M = 0,\ \frac{dx_M}{dt} = 0 \text{ and } \frac{dy_M}{dt} = 83.926 \text{ earth-moon distances/year.}$$

(a) Is this system of ODEs linear or nonlinear?
(b) Convert the system of second order ODES to a system of first order ODEs.
(c) Numerically solve for the dynamic behavior of the moon for 1 year.
(d) According to your model, about how many times does the moon orbit the Earth each year?
(e) Construct a phase plot of x_M and y_M and visually estimate the coordinates of the critical point.
(f) Based on your observed trajectories, describe the type and stability of this critical point. (You do not need to calculate eigenvalues during the exam.)
(g) Holding everything constant except the initial velocity of the moon in the y-direction, solve the problem for a year again for a slower moon with a velocity $\frac{dy_M}{dt} = 52.8$.
(h) What happened in part (g)?
(i) Holding everything constant except the initial velocity of the moon in the y-direction, solve the problem for a year again for a faster moon with a velocity $\frac{dy_M}{dt} = 100.0$.
(j) What happened in part (i)?

Solution:

(a) Is this system of ODEs linear or nonlinear?

This system of ODEs is nonlinear.

(b) Convert the system of second order ODES to a system of first order ODEs.

There is a three-step process to accomplish this transformation. First, identify each new variable.

$$y_1 = x_M$$
$$y_2 = y_M$$
$$y_3 = \frac{dx_M}{dt}$$
$$y_4 = \frac{dy_M}{dt}$$

Second, write the ODE for each new variable.

$$\frac{dy_1}{dt} = y_3$$
$$\frac{dy_2}{dt} = y_4$$
$$\frac{dy_3}{dt} = -\frac{gm_E}{r^3} y_1$$
$$\frac{dy_4}{dt} = -\frac{gm_E}{r^3} y_2$$

where $r = \sqrt{y_1^2 + y_2^2}$

Third, write the ICs for each variable.

$$y_1(t_o) = x_M = 1$$
$$y_2(t_o) = y_M = 0$$
$$y_3(t_o) = \frac{dx_M}{dt} = 0$$
$$y_4(t_o) = \frac{dy_M}{dt} = 83.926$$

(c) Solve for the dynamic behavior of the moon for 1 year.

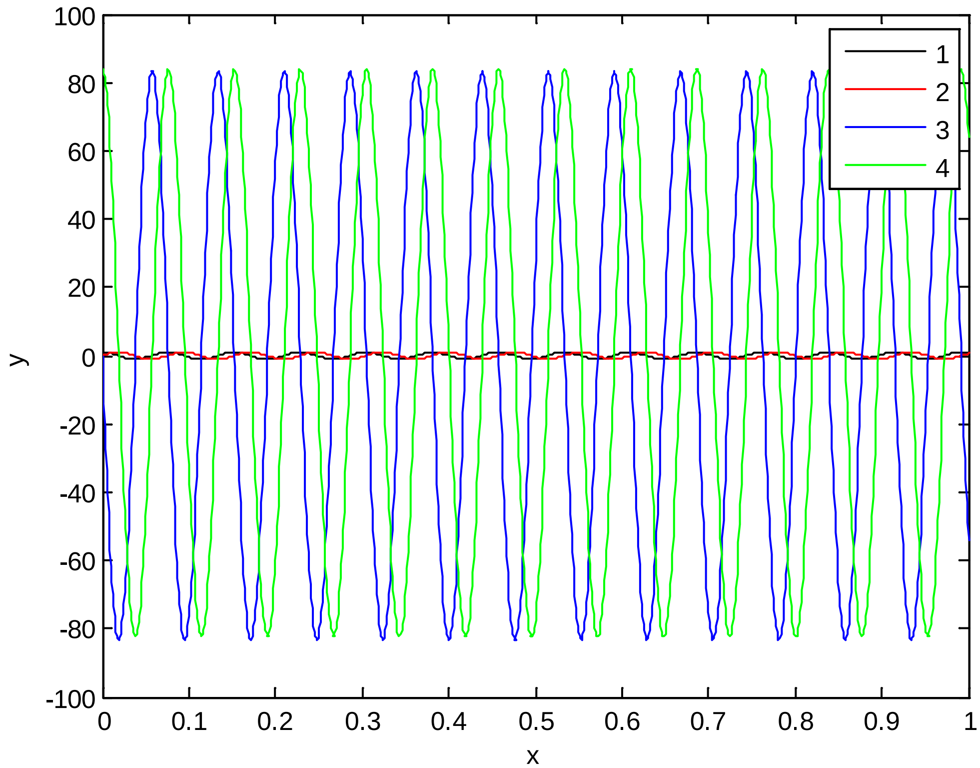
I used the code rk4n.m I modified the input file as follows.

```
function dydx = funkeval(x,y);
m_E = 81.28;
g = 85.8596;
%
x_moon = y(1);
y_moon = y(2);
vx_moon = y(3);
vy_moon = y(4);
%
r = sqrt( x_moon^2 + y_moon^2);
dydx(1) = vx_moon;
dydx(2) = vy_moon;
dydx(3) = -g*m_E/r^3*x_moon;
dydx(4) = -g*m_E/r^3*y_moon;
```

I decided to use discretize a year into 1000 intervals. At the command line prompt, I typed

```
>> [x,y]=rk4n(1000,0,1,[1,0,0,83.926]);
```

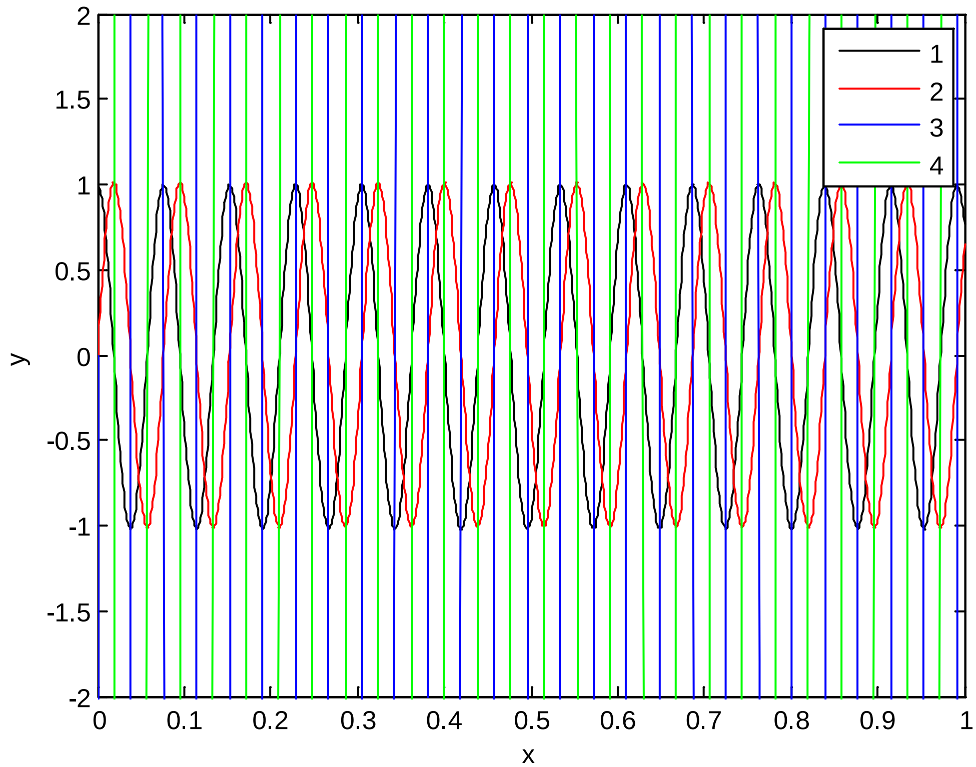
This generated the following plot.



I can view a close up of the positions (variables 1 and 2) by typing the command

```
>> axis([0 1 -2 2])
```

This provides a view of the positions.



(d) According to your model, about how many times does the moon orbit the Earth each year?

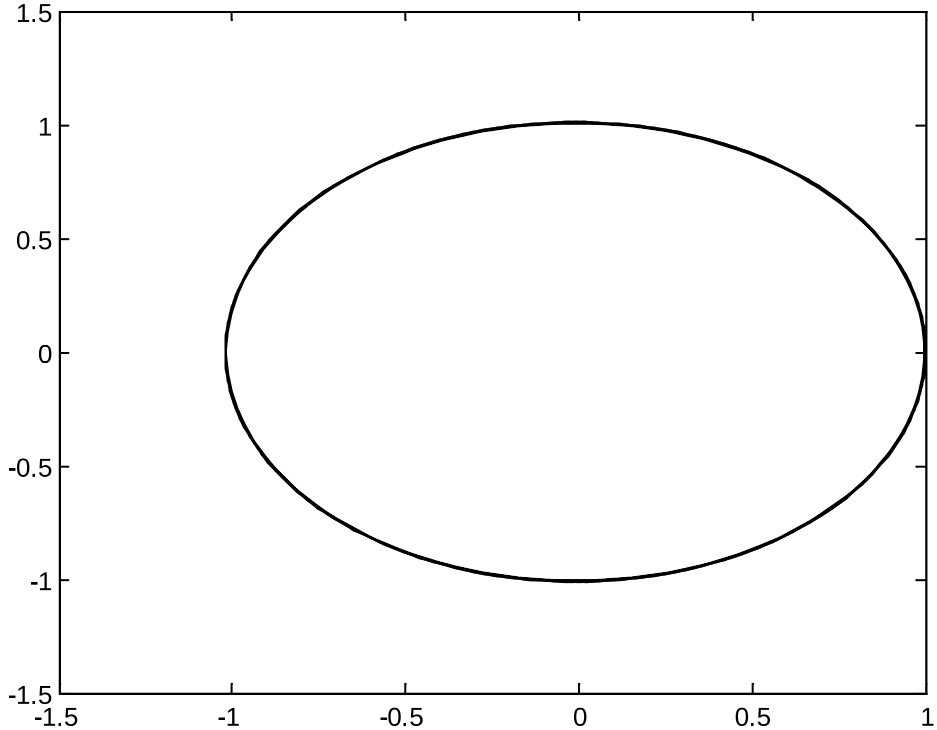
There are a little more than 13 periods in a year according to this plot.

(e) Construct a phase plot of x_M and y_M and visually estimate the coordinates of the critical point.

I constructed a phase plot with the following command.

```
>> figure(2)
>> plot(y(:,1),y(:,2),'k-');
```

This generated the following plot.



Based on a visual inspection of this plot, the critical point is located at the Earth's position at (0,0).

(f) Based on your observed trajectories, describe the type and stability of this critical point.

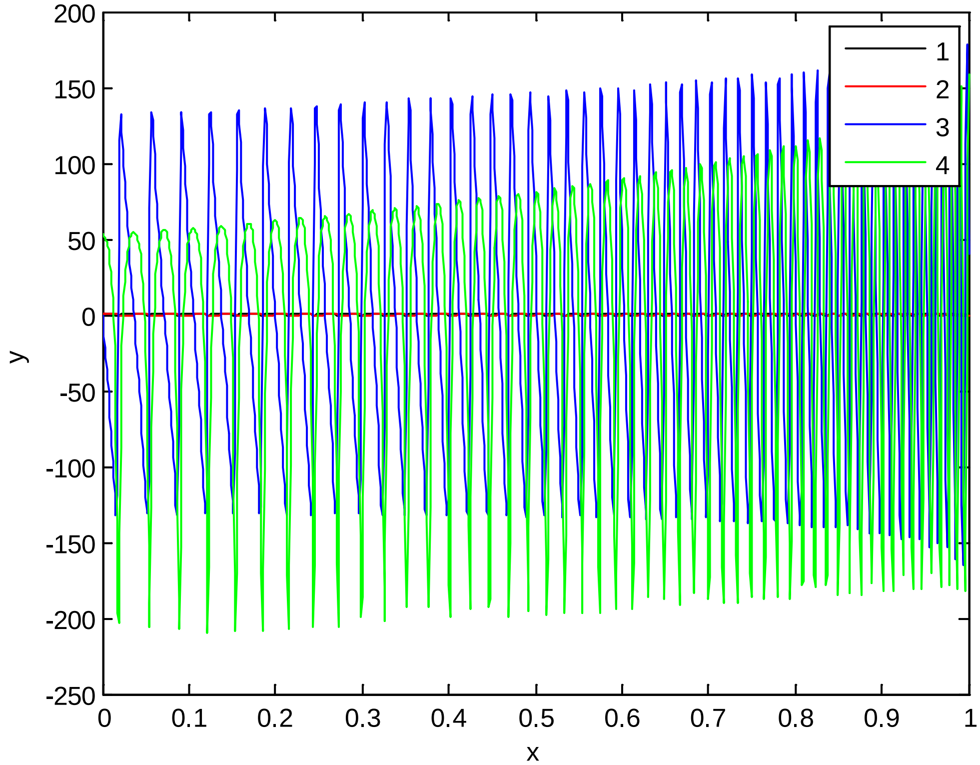
This trajectory is a center and centers are stable. The eigenvalues should be purely imaginary.

(g) Holding everything constant except the initial velocity of the moon in the y-direction, solve the problem for a year again for a slower moon with a velocity $\frac{dy_M}{dt} = 52.8$.

Without making any change to `rk4n.m`, I typed the following command.

```
>> [x,y]=rk4n(1000,0,1,[1,0,0,52.8]);
```

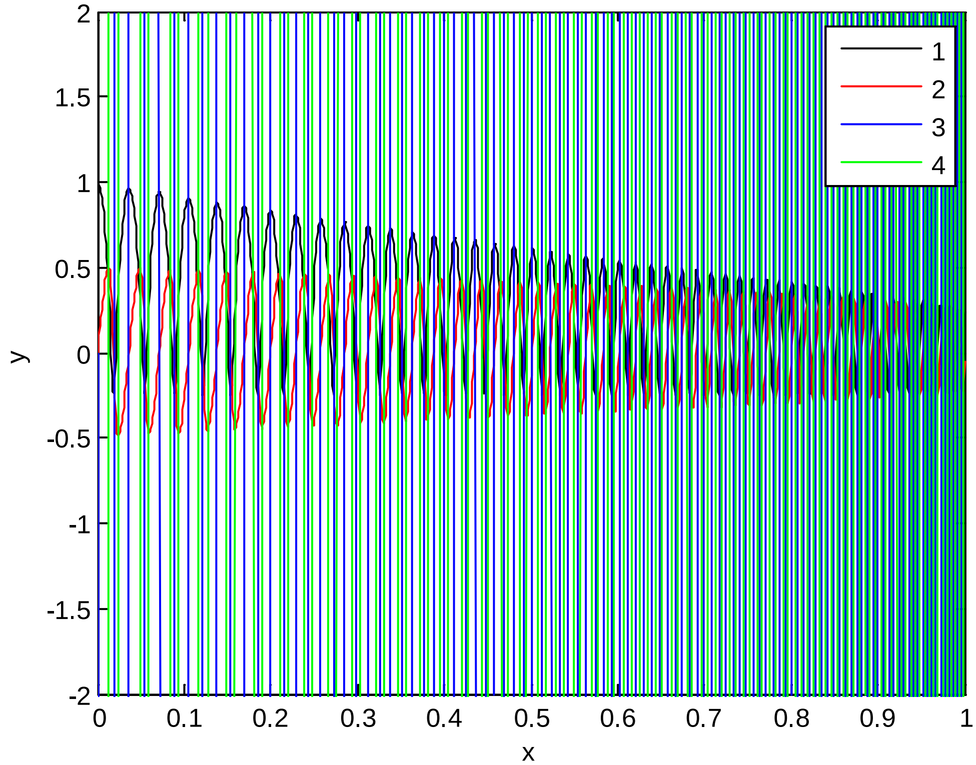
This generated the following plot.



I can view a close up of the positions (variables 1 and 2) by typing the command

```
>> axis([0 1 -2 2])
```

This provides a view of the positions.

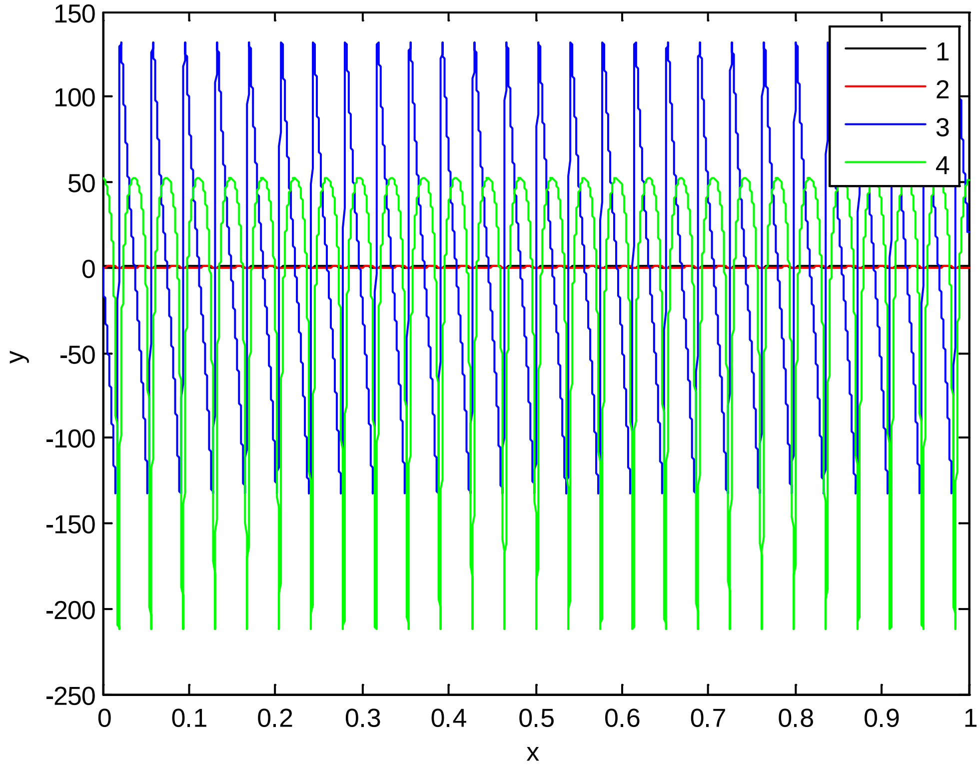


It appears from this plot that the orbit is decaying and the moon will crash into the earth. In order to confirm this behavior, we should run the problem again with a finer temporal resolution, to confirm the behavior.

Without making any change to `rk4n.m`, I typed the following command.

```
>> [x,y]=rk4n(10000,0,1,[1,0,0,52.8]);
```

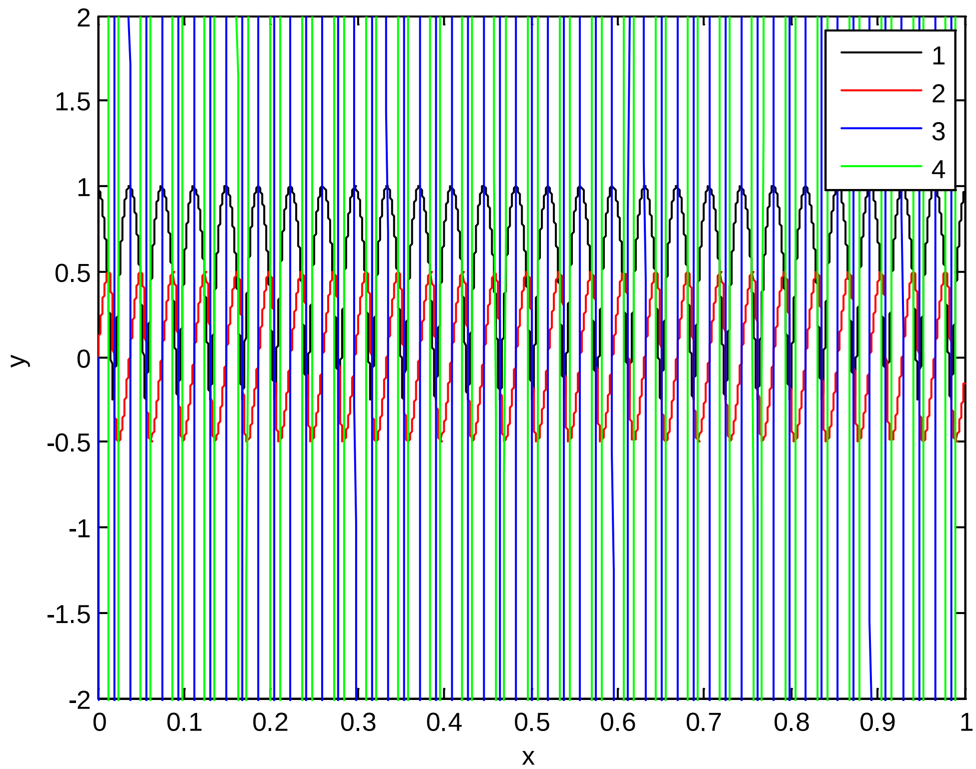
This generated the following plot.



I can view a close up of the positions (variables 1 and 2) by typing the command

```
>> axis([0 1 -2 2])
```

This provides a view of the positions.

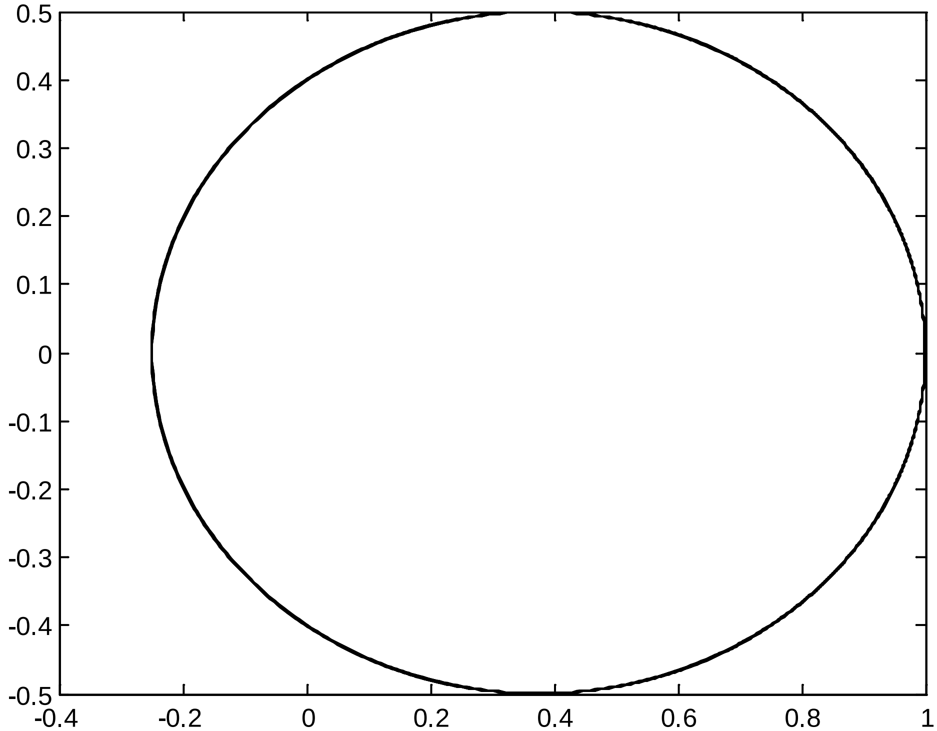


(h) What happened in part (g)?

We can generate a phase plot of the simulation with 10,000 time intervals to more clearly observe the behavior,

```
>> figure(2)
>> plot(y(:,1),y(:,2),'k-');
```

This generated the following plot.

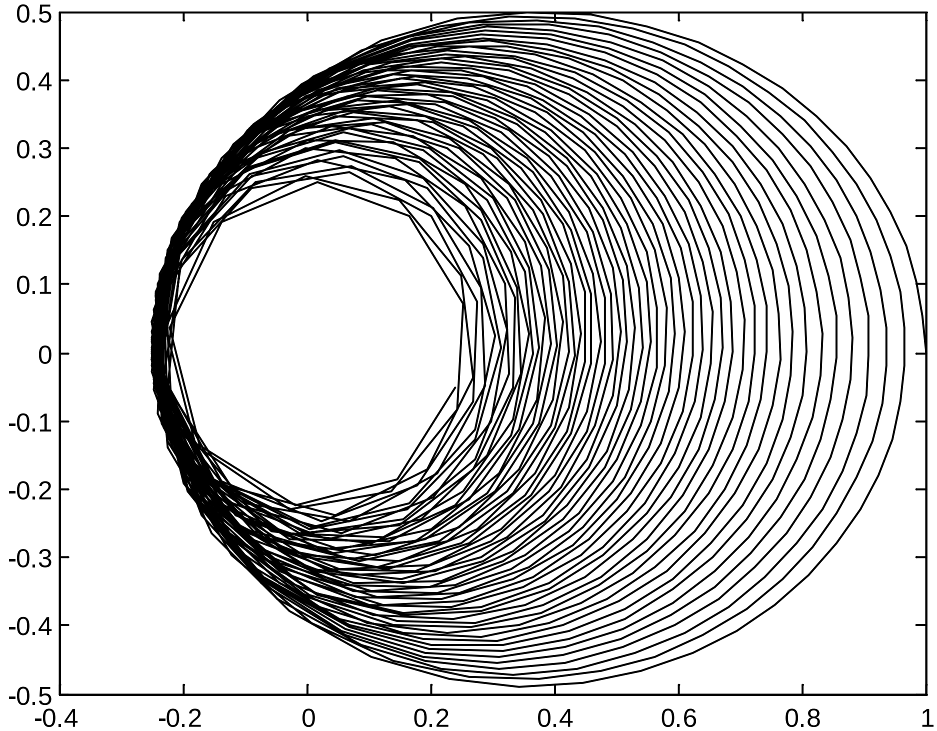


Clearly, we have a stable orbit. In this phase plot, the x values of position range from about -0.2 to 1.0. The y values range from -0.5 to 0.5. So this orbit is elliptical with a smaller range in the y-direction.

From the simulation with only 1,000 intervals, we will obtain an erroneous conclusion. From the plot we can see that x_M and y_M are decaying to zero in an oscillatory manner. A slow moon is crashing into the Earth!
I constructed a phase plot of x_M and y_M with the following command.

```
>> figure(2)
>> plot(y(:,1),y(:,2),'k-');
```

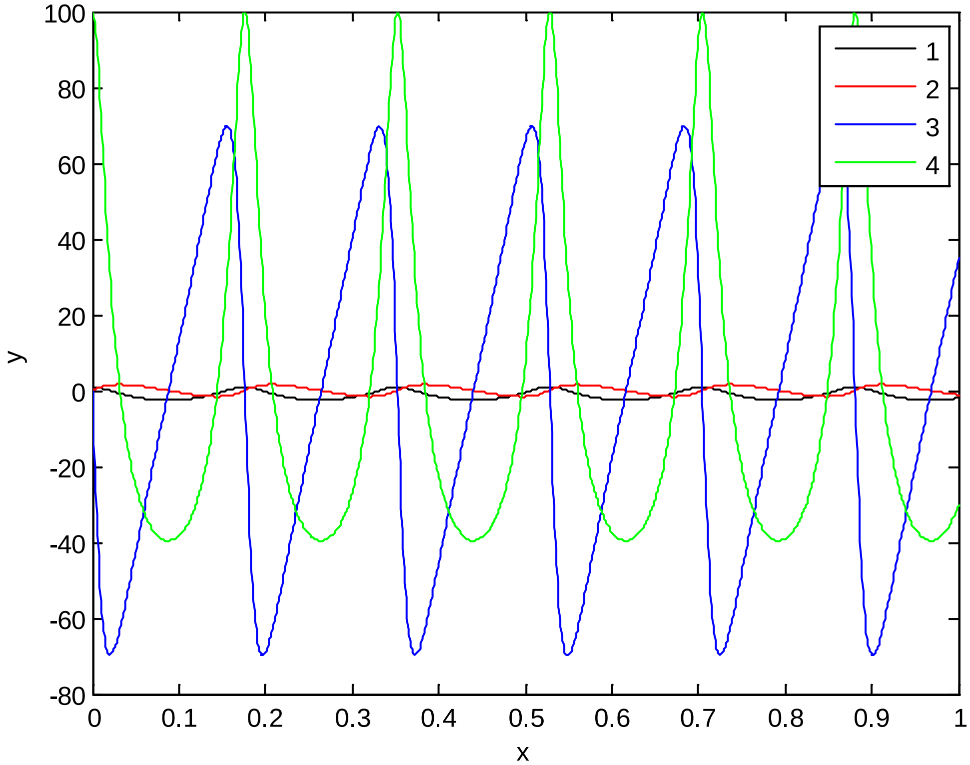
This generated the following plot, in which you can see that the moon is slowly spiraling inward toward the Earth. This aphysical behavior is a consequence of the time step being too large.



(i) Holding everything constant except the initial velocity of the moon in the y-direction, solve the problem for a year again for a faster moon with a velocity $\frac{dy_M}{dt} = 100.0$.
Without making any change to `rk4n.m`, I typed the following command.

```
>> [x,y]=rk4n(1000,0,1,[1,0,0,100.0]);
```

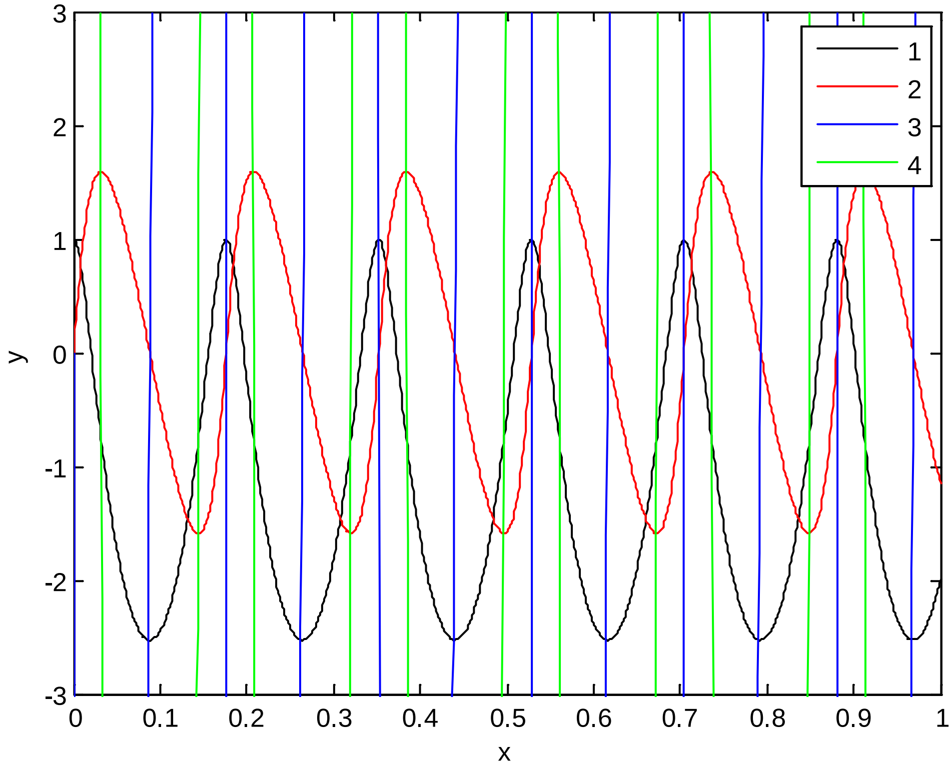
This generated the following plot.



I can view a close up of the positions (variables 1 and 2) by typing the command

```
>> axis([0 1 -3 3])
```

This provides a view of the positions.

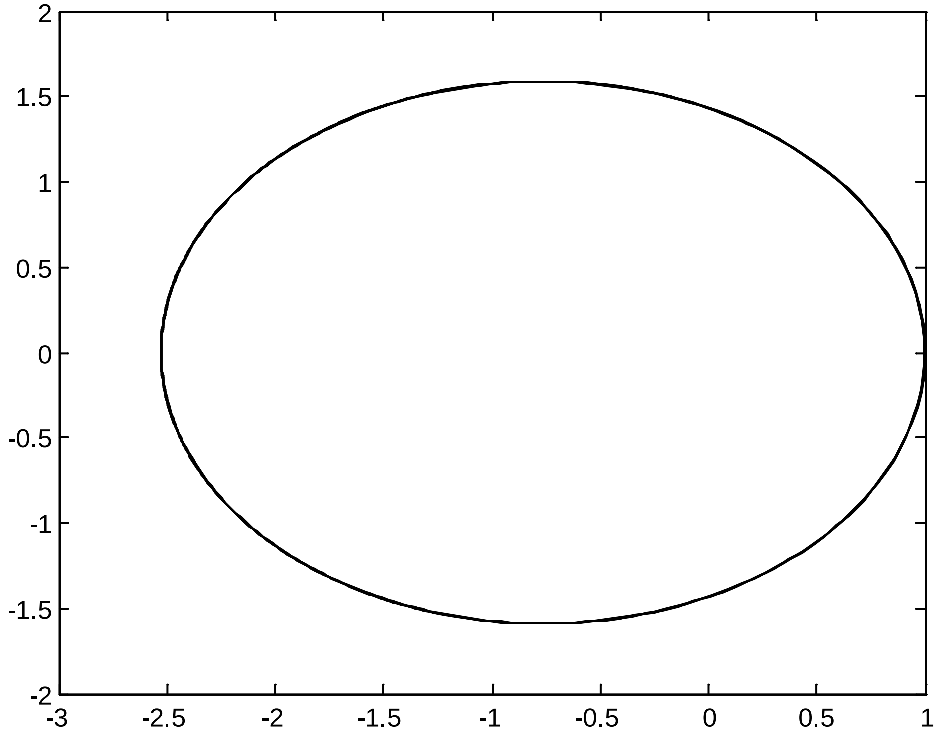


(j) What happened in part (i)?

From the plot we can see that x_M and y_M that the moon has a stable orbit. From the phase plot, shown below, one can see that the orbit is now elliptical.
I constructed a phase plot of x_M and y_M with the following command.

```
>> figure(2)
>> plot(y(:,1),y(:,2),'k-');
```

This generated the following plot.



Here again, you can see that these initial conditions with a fast moon lead to an elliptical orbit. The y position ranges from -1.5 5o 1.5 and the x positions ranges from -2.5 to 1.