ChE 505 Final Exam
Administered: Thursday, December 9, 1999

Problem (1)
Classify the following PDEs in terms of (a) linearity vs nonlinearity (b)hyperbolic, elliptic, or parabolic, and (c) spatial dimensionality.

$$\frac{\partial T}{\partial t} = -D\frac{\partial^2 T}{\partial x^2} + v\frac{\partial T}{\partial x} + k\left(T - T_o\right)$$

$$\frac{\partial^2 T}{\partial t^2} - a\frac{\partial T}{\partial t} = -c\frac{\partial^2 T}{\partial x^2} + k\sqrt{\left(T - T_o\right)}$$

$$0 = -c_x\frac{\partial^2 T}{\partial x^2} - c_y\frac{\partial^2 T}{\partial y^2} - c_z\frac{\partial^2 T}{\partial z^2} + kzT$$

solution:
(a) linear, parabolic, 1-D
(b) nonlinear, hyperbolic, 1-D
(c) linear, elliptic, 3-D

Problem (2)
For each of the three PDEs in problem (1), give a complete set of initial conditions and boundary conditions.

solution:
(a) first order in time: need 1 I.C. second order in space: need 2 B.C.

$$T\left(x, t = t_o\right) = T_0$$
$$T\left(x = x_o, t\right) = T_1$$
$$T\left(x = x_f, t\right) = T_2$$

(a) second order in time: need 2 I.C. second order in space: need 2 B.C. for T and 2 B.C. for dT/dt (because dT/dt appears explicitly in the equation). If dT/dt did not appear explicity in the PDE, then we could get away without B.C.s for dT/dt.

$$T\left(x, t = t_o\right) = T_0$$
$$\frac{\partial T}{\partial t}\left(x, t = t_o\right) = \left(\frac{\partial T}{\partial t}\right)_0$$
$$T\left(x = x_o, t\right) = T_1$$
$$T\left(x = x_f, t\right) = T_2$$
$$\frac{\partial T}{\partial t}\left(x = x_o, t\right) = \left(\frac{\partial T}{\partial t}\right)_1$$
$$\frac{\partial T}{\partial t}\left(x = x_f, t\right) = \left(\frac{\partial T}{\partial t}\right)_2$$

(a) zeroth order in time: need 0 I.C. second order in space: need 2 B.C. for each dimension.

$$T(x = x_o, y, z) = T_1$$
$$T(x = x_f, y, z) = T_2$$
$$T(x, y = y_o, z) = T_3$$
$$T(x, y = y_f, z) = T_4$$
$$T(x, y, z = z_o) = T_5$$
$$T(x, y, z = z_f) = T_6$$

Problem (3)
Describe in words and diagrams and algorithmic flowsheets, how one numerically accounts for a Neumann Boundary Condition of the type:

$$p(x,t)\frac{\partial T}{\partial x} = q(x,t)T + g(x,t)$$

solution:

Specifically see notes for numerical method to solve a single linear parabolic PDE.

Generally, create imaginary node. Use centered finite difference to approximate dT/dx in BC. Use BC to evaluate value of T at imaginary node. Use this value of T in centered finite difference approximations of derivatives appearing in the PDE at the boundary node, adjacent to the imaginary node.

Problem (4)
Classify the following order in terms of (i) linearity vs nonlinearity, (ii) Volterra vs Fredholm, (iii) IE of the first kind vs second kind.

$$\phi(x) = f(x) + \lambda\int_0^x K(x,s)\phi(s)ds$$

solution:

linear in $\phi(x)$

Volterra (upper limit of integration is variable)

second kind, $\phi(x)$ appears both inside and outside integral

Problem (5)
Numerically integrate using Trapezoidal or Simpson's 1/3, use intervals $n_y = n_x = 2$:

$$I = \int_0^1\int_0^y xydxdy$$

solution:

Using trapezoidal rule:

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left\{ \int_a^b f(x,y)dx \right\} dy = \int_c^d \left\{ \frac{b-a}{2n} \left[f(a,y) + f(b,y) + 2\sum_{i=2}^{n} f(x_i, y) \right] \right\} dy = \int_c^d g(y)dy$$

where

$$g(y) = \left\{ \frac{b-a}{2n} \left[f(a,y) + f(b,y) + 2\sum_{i=2}^{n} f(x_i, y) \right] \right\}$$

subject to the fact that the last point we are integrating to is x. When y has only 2 intervals, then we can see that

$$g(y=0) = 0$$

$$g(y=0.5) = \{0.25[f(0,0.5) + f(0.5,0.5)]\} = 0.25(0 + 0.25) = \frac{1}{16}$$

$$g(y=1) = \{0.25[f(0,1) + f(1,1) + 2f(0.5,1)]\} = 0.25(0 + 1 + 1) = \frac{1}{2}$$

Then, applying the trapezoidal rule again:

$$\int_c^d g(y)dy = \frac{d-c}{2n} \left[g(c) + g(d) + 2\sum_{j=2}^{n} g(y_j) \right]$$

$$\int_c^d g(y)dy = 0.25 \left[0 + \frac{1}{2} + 2\frac{1}{16} \right] = \frac{5}{32}$$

Problem (6)
Consider water flowing through a horizontal pipe that is initially empty. If there is not enough flow to fill the pipe cross-section with water, set up the equations to describe the transient flow-system.

solution:

Not posted.
Highly dependent on the assumptions made.