ChE/MSE 505
Midterm Examination
Administered: Monday, October 11, 2006

The general momentum balance,

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho \mathbf{v} \cdot \nabla(\mathbf{v}) - \nabla p - \nabla \cdot \boldsymbol{\tau} - \rho \nabla \hat{\Phi} \qquad (1)$$

under the following assumptions: (1) incompressible flow, (2) one-dimensional system, (3) steady state, (4) external fields of $\nabla \hat{\Phi} = \frac{v - v_o}{\tau}$, (5) negligible pressure gradient, (6) Newtonian fluid, and (7) isothermal flow reduces to

$$0 = -\rho v \frac{\partial v}{\partial z} + \eta \frac{d^2 v}{dz^2} - \rho \frac{v - v_o}{\tau} \qquad (2)$$

where v is velocity in the z direction, ρ is density, η is an elongational viscosity, and v_o is constant of the external field and τ is a strictly positive constant of the external field.

Answer the following questions and perform the following tasks.

1. Is equation (2) an ODE or PDE?

ODE

2. Is equation (2) linear or nonlinear?

nonlinear

3. Convert the second order DE equation in (2) to a system of first-order DEs.

The variable transformation is

$$y_1 = v$$

$$y_2 = \frac{\partial v}{\partial z}$$

The first-order ODEs are

$$\frac{\partial y_1}{\partial z} = y_2$$

$$\frac{dy_2}{dz} = \frac{\rho}{\eta} y_1 y_2 + \rho \frac{y_1 - v_o}{\tau\eta}$$

4. Calculate the critical point of the system

At the the time derivatives are zero.

From the first equation:

$$y_2 = 0$$

From the second equation:

$$y_1 = v_o$$

5. Calculate the eigenvalues of this equation at the critical point.

linear the system of equations

$$\frac{\partial \underline{y}}{\partial z} \approx \underline{\underline{J}}\,\underline{y} + \underline{R}$$

The Jacobian is

$$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ \frac{\rho}{\eta} y_2 + \frac{\rho}{\tau\eta} & \frac{\rho}{\eta} y_1 \end{bmatrix}$$

The Jacobian evaluated at the critical point is

$$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ \frac{\rho}{\tau\eta} & \frac{\rho}{\eta} v_o \end{bmatrix}$$

The eigenvalues of the Jacobian are obtained from the characteristic equation

$$\det\left(\underline{\underline{J}} - \lambda \underline{\underline{I}}\right) = (-\lambda)\left(\frac{\rho v_o}{\eta} - \lambda\right) - \frac{\rho}{\tau\eta} = 0$$

simplify

$$\det\left(\underline{\underline{J}} - \lambda \underline{\underline{I}}\right) = \lambda^2 - \lambda \frac{\rho v_o}{\eta} - \frac{\rho}{\tau\eta} = 0$$

solve using the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{\rho v_o}{\eta} \pm \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}}}{2}$$

6. Determine the type and stability of the critical point.

The density and viscosity are always positive. The constant τ is defined in the problem statement to be strictly positive. Therefore everything inside the square root is positive. As a result, the eigenvalues are purely real. Therefore, the type of our critical point is either an improper node or a saddle point depending upon the signs of the eigenvalues.

Since $4\frac{\rho}{\tau\eta} > 0$, we know that

$$4\frac{\rho}{\tau\eta} + \left(\frac{\rho v_o}{\eta}\right)^2 > \left(\frac{\rho v_o}{\eta}\right)^2 \tag{3}$$

case 1. Now let's consider when v_o is positive. Since both sides of this inequality are positive, we also know that $\sqrt{4\frac{\rho}{\tau\eta} + \left(\frac{\rho v_o}{\eta}\right)^2} > \frac{\rho v_o}{\eta}$. If this is true, then we know that

$$\frac{\rho v_o}{\eta} + \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}} > 0 \quad \text{and} \quad \frac{\rho v_o}{\eta} - \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}} < 0 .$$

So that

$$\lambda_1 = \frac{\frac{\rho v_o}{\eta} + \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}}}{2} > 0 \quad \text{and} \quad \lambda_2 = \frac{\frac{\rho v_o}{\eta} - \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}}}{2} < 0 .$$

In this case, one eigenvalue is positive and the other is negative and we have a saddle point. A saddle point is always unstable.

case 2. Now let's consider when v_o is negative. We now consider the inequality in (3) when we take the square root, which become s $\sqrt{4\frac{\rho}{\tau\eta} + \left(\frac{\rho v_o}{\eta}\right)^2} > \left|\frac{\rho v_o}{\eta}\right|$. If this is true, then we know that

$$\frac{\rho v_o}{\eta} + \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}} > 0 \quad \text{and} \quad \frac{\rho v_o}{\eta} - \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}} < 0 .$$

So that

$$\lambda_1 = \frac{\frac{\rho v_o}{\eta} + \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}}}{2} > 0 \qquad \text{and} \qquad \lambda_2 = \frac{\frac{\rho v_o}{\eta} - \sqrt{\left(\frac{\rho v_o}{\eta}\right)^2 + 4\frac{\rho}{\tau\eta}}}{2} < 0 .$$

In this case, one eigenvalue is positive and the other is negative and we have a saddle point. A saddle point is always unstable.