ChE/MSE 505
Advanced Mathematic for Engineers
Final Exam
Fall Semester, 2004
Instructor: David Keffer
Administered: 8:00-10:00 am, Wednesday December 8, 2004

Problem 1.
Consider the partial differential equation describing the steady-state temperature profile in an uninsulated cylindrical metal rod of radius *R* and length *L*, in which there is variation in the radial (*r*) and axial (*z*) directions only. The two axial ends of the rod are held at fixed temperatures.

$$0 = \alpha \nabla^2 T \tag{1}$$

where α is the thermal diffusivity. The Laplacian in two-dimensional cylindrical coordinates is defined as

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} . \tag{2}$$

You may need the following properties: h is the heat transfer coefficient, ρ is the density, C_p is the heat capacity, k_c is the thermal conductivity, and T_{surr} is the temperature of the surroundings.

(a) Categorize the type of PDE: parabolic, hyperbolic, or elliptic.
(b) Determine the linearity of the PDE.
(c) Specify a complete set of consistent initial and boundary conditions for this problem.
(d) Identify and provide a detailed algorithm describing a numerical technique suitable for the solution of this problem.

Solution:

(a) Categorize the type of PDE: parabolic, hyperbolic, or elliptic.

The PDE is elliptic. There is no time derivative of any order in it.

(b) Determine the linearity of the PDE.

The PDE is linear in the unknown, T.

(c) Specify a complete set of consistent initial and boundary conditions for this problem.

There are no initial conditions for an elliptic PDE.

A complete set of consistent boundary conditions are

$T(z=0,r)=T_o$ (Temperature is fixed at axial end 1.)

$T(z=L,r)=T_f$ (Temperature is fixed at axial end 1.)

$\left.\frac{\partial T}{\partial r}\right|_{r=o}=0$ (Radial symmetry.)

$\frac{q}{A}=h\left(T(z,r=R)-T_{surr}\right)=-k_c\left.\frac{\partial T}{\partial r}\right|_{r=R}$ (Uninsulated boundary, flux is heat lost.)

(d) Identify and provide a detailed algorithm describing a numerical technique suitable for the solution of this problem.

For solving a linear elliptic PDE, we first discretize *r* and *z* directions. Then for each point in *r*,*z* space, we can write an equation of the form:

$$0=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}$$

$$0=\left(\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta r^2}\right)+\frac{1}{r_{i,j}}\left(\frac{T_{i+1,j}-T_{i-1,j}}{\Delta r}\right)+\left(\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta z^2}\right).$$

If we have n_r intervals in the *r* direction and n_z intervals in the *z* direction, then we are going to have $n_r n_z$ equations of this form. They will comprise a system of linear algebraic equations. We can solve them using linear algebra.

Problem 2.
If the cylinder in problem 1 also loses heat to the surroundings via radiation, we must add an additional term to the energy balance in equation (1),

$$0=\alpha\nabla^2 T-\frac{2\varepsilon\sigma}{R\rho C_p}\left(T^4-T_{surr}^4\right) \quad (3)$$

where ε is the total emissivity of the cylinder and σ is a proportionality constant.
(e) Determine the linearity of the PDE.
(f) Identify and provide a detailed algorithm describing a numerical technique suitable for the solution of this problem.

Solution

(e) Determine the linearity of the PDE.
The PDE is now nonlinear.
(f) Identify and provide a detailed algorithm describing a numerical technique suitable for the solution of this problem.

For solving a nonlinear elliptic PDE, we first discretize *r* and *z* directions as we did for the linear case. Then for each point in *r*,*z* space, we can write an equation of the form:

$$0=\left(\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta r^2}\right)+\frac{1}{r_{i,j}}\left(\frac{T_{i+1,j}-T_{i-1,j}}{\Delta r}\right)+\left(\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta z^2}\right)-\frac{2\varepsilon\sigma}{\alpha R\rho C_p}\left(T_{i,j}^4-T_{surr}^4\right)$$

If we have n_r intervals in the *r* direction and n_z intervals in the *z* direction, then we are going to have $n_r n_z$ equations of this form. They will comprise a system of nonlinear algebraic equations. We can solve them using the Newton-Raphson method. If we have a hard time getting an initial guess, we can insert a parameter, ζ in front of the radiation term. We can parameter step through ζ from 0 to 1, using the converged solution from the previous value of ζ as an initial guess for the next value of ζ. The first initial guess will come from the solution of the linear problem when $\zeta = 0$.

Problem 3.
We want to use the following equation to fit some vapor pressure data.

$$P^{vap}=\exp\left(\frac{A}{B+T}\right) \tag{4}$$

where *T* is temperature and *A* and *B* are fitting constants. We have two pieces of data: the vapor pressure at 300 K is 1.1 atm and the vapor pressure at 320 K is 1.7 atm. Given this experimental data find the best values of A and B.

Solution:

We have two unknowns *A* and *B*. Our life would be much simpler if we can rearrange the problem so that it is linear in *A* and *B*. Let's try.

$$(B+T)\ln\left(P^{vap}\right)-A=0$$

There we have it. The equation is linear in *A* and *B*. We have a system of two linear equations and two unknowns.

$$f_1(A,B)=(B+T_1)\ln\left(P_1^{vap}\right)-A$$
$$f_2(A,B)=(B+T_2)\ln\left(P_2^{vap}\right)-A$$

We can write this in matrix notation as

$$\underline{\underline{\mathbf{J}}}\,\underline{\mathbf{x}}=\underline{\mathbf{R}}$$

where

$$\underline{\underline{\mathbf{J}}} = \begin{bmatrix} -1 & \ln\left(P_1^{vap}\right) \\ -1 & \ln\left(P_2^{vap}\right) \end{bmatrix}, \ \underline{\mathbf{x}} = \begin{bmatrix} A \\ B \end{bmatrix}, \text{ and } \underline{\mathbf{R}} = \begin{bmatrix} -T_1 \ln\left(P_1^{vap}\right) \\ -T_2 \ln\left(P_2^{vap}\right) \end{bmatrix}$$

We need the determinant and inverse

$$\det\left(\underline{\underline{\mathbf{J}}}\right) = -\ln\left(P_2^{vap}\right) + \ln\left(P_1^{vap}\right) = \ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)$$

$$\underline{\underline{\mathbf{J}}}^{-1} = \frac{1}{\det\left(\underline{\underline{\mathbf{J}}}\right)} \begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{\ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)} \begin{bmatrix} \ln\left(P_2^{vap}\right) & -\ln\left(P_1^{vap}\right) \\ 1 & -1 \end{bmatrix}$$

The solution is given by

$$\underline{\mathbf{x}} = \underline{\underline{\mathbf{J}}}^{-1}\underline{\mathbf{R}} = \frac{1}{\ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)} \begin{bmatrix} \ln\left(P_2^{vap}\right) & -\ln\left(P_1^{vap}\right) \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -T_1 \ln\left(P_1^{vap}\right) \\ -T_2 \ln\left(P_2^{vap}\right) \end{bmatrix} = \frac{1}{\ln\left(\frac{P_1^{vap}}{P_2^{vap}}\right)} \begin{bmatrix} \left(T_2 - T_1\right)\ln\left(P_1^{vap}\right)\ln\left(P_2^{vap}\right) \\ -T_1 \ln\left(P_1^{vap}\right) + T_2 \ln\left(P_2^{vap}\right) \end{bmatrix}$$

We have two pieces of data: the vapor pressure at 300 K is 1.1 atm and the vapor pressure at 320 K is 1.7 atm.

$$\underline{\mathbf{x}} = \frac{1}{\ln\left(\frac{1.1}{1.7}\right)} \begin{bmatrix} 20\ln(1.1)\ln(1.7) \\ -300\ln(1.1) + 320\ln(1.7) \end{bmatrix} \approx -2.2972 \begin{bmatrix} 1.0115 \\ 141.21 \end{bmatrix} = \begin{bmatrix} -2.3236 \\ -324.38 \end{bmatrix}$$