Midterm Examination
Administered: Monday, October 11, 2004

Consider the second-order nonlinear ordinary differential equation boundary value problem.

$$\frac{d^2y}{dt^2} + \sin\left(\frac{d^2y}{dt^2}\right) = \frac{dy}{dt} - y \tag{1}$$

with the boundary conditions

$$y(t_o) = y_o \quad \text{and} \quad y(t_f) = y_f \tag{2}$$

Provide a detailed step-by-step algorithm of how you numerically obtain an approximate solution to this problem. Write the equations for specific examples of algorithms, e.g. Newton-Raphson or Euler, where used. Indicate loops where necessary.

solution:

In class, we discussed nesting an algebraic equation (AE) solver, like the Newton Raphson method, inside an ODE solver, like the Runge Kutta method, when we couldn't isolate the derivative on the left-hand-side of the ODE. In class we also separately discussed nesting an ODE solver inside an AE solver to solve boundary value problems. This problem combines the two cases. We have a boundary value problem in which the derivative cannot be isolated on the left-hand-side. For this problem, we will have to nest an AE solver inside an ODE solver inside an AE solver.

First, this ODE is a second order ODE, so transform it to a system of two first order ODEs.

Transformation step 1. Define variables.

$$y_1 = y \tag{3.a}$$

$$y_2 = \frac{dy}{dt} \tag{3.b}$$

Transformation step 2. Write first-order ODEs.

$$\frac{dy_1}{dt} = y_2 \tag{4.a}$$

$$\frac{dy_2}{dt} + \sin\left(\frac{dy_2}{dt}\right) = y_2 - y_1 \tag{4.b}$$

Transformation step 3. Write conditions.

$$y_1(t_o) = y_o \quad \text{and} \quad y_1(t_f) = y_f \tag{5}$$

Now the transformation is complete. We can write an algorithm to solve this system of first order ODEs.

The solution has three loops.

The outermost loop is an iterative loop to determine the correct initial value of $y_2(t_o)$. This value is required so that we can solve the system of ODEs as an initial value problem. The outermost loop would be, for example, a Newton-Raphson iterative loop.

The middle loop is a standard ODE-solving root, like the Euler method (or RK4, if we want to be more accurate). Given a guess of $y_2(t_o)$, we can solve the ODEs out to time t_f. Then we can compare our solution with the boundary condition at t_f to determine if we have a good guess of $y_2(t_o)$.

The interior loop is another iterative loop because for every time step in the middle loop, we require the value of $\frac{dy_2}{dt}$. However, equation (4.b) is nonlinear in $\frac{dy_2}{dt}$. We cannot isolate $\frac{dy_2}{dt}$. Therefore, we need to use something like Newton-Raphson method to solve for $\frac{dy_2}{dt}$ at each timestep.

The specific algorithm is as follows.

1. Guess a value of $y_2(t_o)$.

2. Guess a value of $\left.\frac{dy_2}{dt}\right|_{t_o}$

3. Use an iterative procedure to solve nonlinear algebraic equations, such as the Newton-Raphson method with numerical approximations to the deriviatives, to solve equation (4.b) at t_o. Start with the ODE for y_2 evaluated at t_o.

$$\left.\frac{dy_2}{dt}\right|_{t_o} + \sin\left(\left.\frac{dy_2}{dt}\right|_{t_o}\right) = y_2(t_o) - y_1(t_o)$$

3.a We can rearrange and write this equation as

$$g\left(\left.\frac{dy_2}{dt}\right|_{t_o}\right) = \left.\frac{dy_2}{dt}\right|_{t_o} + \sin\left(\left.\frac{dy_2}{dt}\right|_{t_o}\right) - y_2(t_o) + y_1(t_o) \quad (6)$$

3.b Evaluate equation (6) at the current time.

3.c Evaluate the derivative of equation (6), $\frac{dg}{d\left(\left.\frac{dy_2}{dt}\right|_{t_o}\right)}$ by evaluating equation (6) at $\left.\frac{dy_2}{dt}\right|_{t_o}+h$ and $\left.\frac{dy_2}{dt}\right|_{t_o}-h$, so that $\frac{dg}{d\left(\left.\frac{dy_2}{dt}\right|_{t_o}\right)} \approx \frac{g\left(\left.\frac{dy_2}{dt}\right|_{t_o}+h\right) - g\left(\left.\frac{dy_2}{dt}\right|_{t_o}-h\right)}{2h}$.

3.d Evaluate a new guess for $\left.\frac{dy_2}{dt}\right|_{t_o}$, using the Newton-Raphson formula.

$$\left.\frac{dy_2}{dt}\right|_{t_o}^{new} = \left.\frac{dy_2}{dt}\right|_{t_o}^{old} - \frac{g\left(\left.\frac{dy_2}{dt}\right|_{t_o}\right)}{\frac{dg}{d\left.\frac{dy_2}{dt}\right|_{t_o}}}$$

3.e. Using this new value of $\left.\frac{dy_2}{dt}\right|_{t_o}^{new}$, loop back to step (3.b) until the residual in equation (6) is within an acceptably small tolerance.

4. Now that we have the time derivatives at the current time, use the Euler method, for example, to solve for your variables at the new time:

$$y_1(t_1) = y_1(t_o) + \Delta t \left.\frac{dy_1}{dt}\right|_{t_o} = y_1(t_o) + \Delta t y_2(t_o)$$

$$y_2(t_1) = y_2(t_o) + \Delta t \left.\frac{dy_2}{dt}\right|_{t_o}$$

5. Loop back to step 3, now solving for $\left.\frac{dy_2}{dt}\right|_{t_{i+1}}$ at the next time by equation equation (4.b) at t_{i+1}. Exit from this loop when you have reached t_f. At the next time step, use your converged value of $\left.\frac{dy_2}{dt}\right|_{t_i}$ as the initial guess for $\left.\frac{dy_2}{dt}\right|_{t_{i+1}}$.

6. Calculate the residual of the outer most loop, namely

$$f(y_2(t_o)) = y_1(t_f) - y_f \tag{7}$$

where $y_1(t_f)$ was obtained in step 5 and y_f is the given boundary condition.

7. Calculate the numerical approximation of the derivative, $\frac{df}{dy_2(t_o)}$, by looping through steps 2 through 5 for $y_2(t_o)+h$ and $y_2(t_o)-h$, so that $\frac{df}{dy_2(t_o)} \approx \frac{f(y_2(t_o)+h) - f(y_2(t_o)-h)}{2h}$.

8. Use the Newton Raphson method to estimate a new value of $y_2(t_o)$

$$y_2(t_o)^{new} = y_2(t_o)^{old} - \frac{f\left(y_2(t_o)^{old}\right)}{\frac{df}{dy_2(t_o)}}$$

9. Using this new value of $y_2(t_o)^{new}$, loop back to step (2) until the residual in equation (7) is within an acceptably small tolerance.