Exam I
Administered: Monday, February 5, 2001
24 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (4 points)

Consider the PDF

$$f(x,y) = c \cdot {}_xP_y \qquad \text{for } x = 1, 2, \text{ and } 3 \text{ and } y = 1 \text{ to } x$$

(a) find the value of c that makes this PDF legitimate.
(b) find $P(y = 1 | x = 2)$

Solution:

This problem is discrete in both x and y because the permutation can only take on integer values. Therefore y can take on integer values from 1 to x. (You cannot evaluate ${}_xP_y$ for y > x. How can you select y=3 objects, when you only have x=2 objects?)

(a) find the value of c that makes this PDF legitimate.

$$f(1,1) = c \cdot {}_1P_1 = c \qquad f(3,1) = c \cdot {}_3P_1 = 3c$$
$$f(2,1) = c \cdot {}_2P_1 = 2c \qquad f(3,2) = c \cdot {}_3P_2 = 6c$$
$$f(2,2) = c \cdot {}_2P_2 = 2c \qquad f(3,3) = c \cdot {}_3P_3 = 6c$$

$$\sum_{x=1}^{3}\sum_{y=1}^{x} f(x,y) = 1$$
$$\sum_{x=1}^{3}\sum_{y=1}^{x} f(x,y) = 20c = 1$$

$$c = \frac{1}{20}$$

(b) find $P(y = 1 | x = 2)$

The marginal distribution of x is

$$g(x) = \sum_{y=1}^{x} f(x,y)$$

$$g(2) = \sum_{y=1}^{x} f(2,y) = f(2,1) + f(2,2) = \frac{2}{20} + \frac{2}{20} = \frac{1}{5}$$

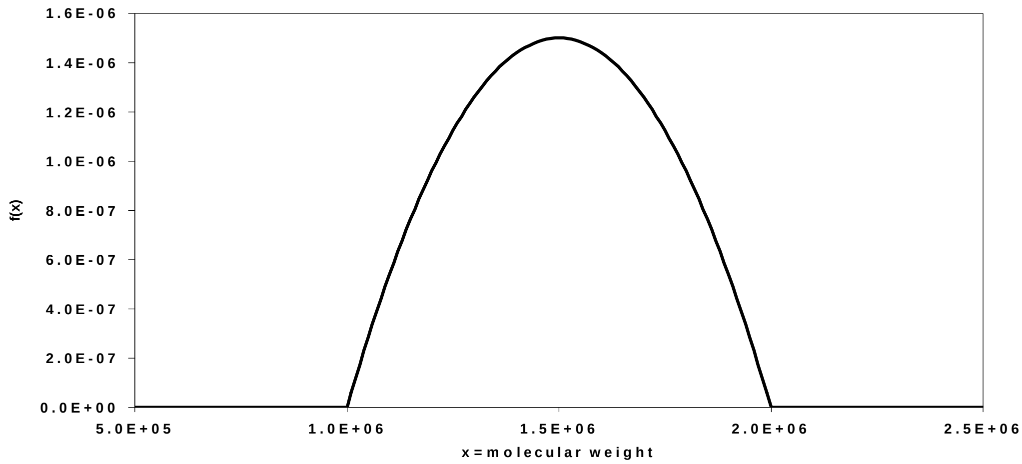
$$P(y=1\,|\,x=2)=\frac{f(x=2,y=1)}{g(x=2)}=\frac{2/20}{1/5}=\frac{1}{2}$$

Problem 2. (10 points)

An experimental polymer reactor creates polyethylene with a molecular weight distribution of

$$f(x)=\begin{cases}-1.2\cdot10^{-5}+1.8\cdot10^{-11}x-6.0\cdot10^{-18}x^2 & 1\cdot10^6\le x\le 2\cdot10^6\\ 0 & \text{otherwise}\end{cases}$$

where x is the molecular weight. A plot of f(x) is shown below.



(a) Is this problem continuous or discrete?
(b) What is the probability of obtaining a polymer with molecular weight in the range $5\cdot10^5\le x\le 1.25\cdot10^6$?
(c) What is the probability of obtaining a polymer with molecular weight greater than $1.25\cdot10^6$?
(d) What is the mean molecular weight?
(e) The compressibility of the polymer at a certain temperature is given by

$$\kappa=6.5\cdot10^{-6}x-0.59$$

What is the mean of the compressibility?

Solution:
(a) Is this problem continuous or discrete?

This problem is continuous because, for the given PDF, the random variable can assume any real value between $1\cdot10^6\le x\le 2\cdot10^6$.

(b) What is the probability of obtaining a polymer with molecular weight in the range $5\cdot10^5\le x\le 1.25\cdot10^6$?

The PDF has a different definition from $5 \cdot 10^5 \le x \le 1.0 \cdot 10^6$ than from $1.0 \cdot 10^6 \le x \le 1.25 \cdot 10^6$ so you must break up the integral into two parts.

$$P(5 \cdot 10^5 \le x \le 1.25 \cdot 10^6) = \int_{5 \cdot 10^5}^{1.00 \cdot 10^6} f(x)dx + \int_{1.00 \cdot 10^6}^{1.25 \cdot 10^6} f(x)dx = 0 + \int_{1.00 \cdot 10^6}^{1.25 \cdot 10^6} f(x)dx$$

$$P(5 \cdot 10^5 \le x \le 1.25 \cdot 10^6) = \int_{1.00 \cdot 10^6}^{1.25 \cdot 10^6} \left(-1.2 \cdot 10^{-5} + 1.8 \cdot 10^{-11} x - 6.0 \cdot 10^{-18} x^2\right) dx$$

$$P(5 \cdot 10^5 \le x \le 1.25 \cdot 10^6) = \left(-1.2 \cdot 10^{-5} x + \frac{1.8 \cdot 10^{-11}}{2} x^2 - \frac{6.0 \cdot 10^{-18}}{3} x^3\right)\Bigg|_{1.00 \cdot 10^6}^{1.25 \cdot 10^6}$$

$$P(5 \cdot 10^5 \le x \le 1.25 \cdot 10^6) = (-4.84375) - (-5) = 0.15625$$

(c) What is the probability of obtaining a polymer with molecular weight greater than $1.25 \cdot 10^6$?

If you use the definition, you will obtain the correct answer but this requires you to perform the integration again. Instead, use the fact that the probabilities must sum to one.

$$P(x \le 1.25 \cdot 10^6) + P(x > 1.25 \cdot 10^6) = 1$$

$$P(x > 1.25 \cdot 10^6) = 1 - P(x \le 1.25 \cdot 10^6) = 1 - 0.15625 = 0.84375$$

(d) What is the mean molecular weight?

By examining the plot, you can see that the molecular weight distribution is symmetric about x $= 1.5 \cdot 10^6$. Therefore, the mean is $\mu_x = 1.5 \cdot 10^6$.

However, if you don't see the symmetry, you can always use the definition of the mean.

$$\mu_x = \int_{1.0 \cdot 10^6}^{2.0 \cdot 10^6} x f(x)dx = \int_{1.0 \cdot 10^6}^{2.0 \cdot 10^6} x\left(-1.2 \cdot 10^{-5} + 1.8 \cdot 10^{-11} x - 6.0 \cdot 10^{-18} x^2\right) dx$$

$$\mu_x = \int_{1.0 \cdot 10^6}^{2.0 \cdot 10^6} \left(-1.2 \cdot 10^{-5} x + 1.8 \cdot 10^{-11} x^2 - 6.0 \cdot 10^{-18} x^3\right) dx$$

$$\mu_x = \left(-\frac{1.2 \cdot 10^{-5}}{2} x^2 + \frac{1.8 \cdot 10^{-11}}{3} x^3 - \frac{6.0 \cdot 10^{-18}}{4} x^4\right)\Bigg|_{1.0 \cdot 10^6}^{2.0 \cdot 10^6}$$

$$\mu_x = 0 - \left(-1.5 \cdot 10^6\right) = 1.5 \cdot 10^6$$

(e) The compressibility of the polymer at a certain temperature is given by

$$\kappa = 6.5 \cdot 10^{-6} x - 0.59$$

What is the mean of the compressibility?

You can use the definition of the mean to obtain the mean compressibility or you can take advantage of the fact that the mean is a linear operator:

$$\mu_\kappa = \mu_{6.5 \cdot 10^{-6} x - 0.59} = 6.5 \cdot 10^{-6} \mu_x - 0.59 = 6.5 \cdot 10^{-6}\left(1.5 \cdot 10^{6}\right) - 0.59 = 9.16$$

Problem 3. (10 points)

You are monitoring a process using a packed bed of 2 adsorbents (zeolite-X & zeolite-Y) to adsorb methane and ethane. The zeolite-X in the bed adsorbs 70% of the molecules adsorbed. The probability that an adsorbed molecule is methane in zeolite X is 0.08. The probability that a molecule is ethane given that it is adsorbed in zeolite-Y is 0.8537. Answer the following questions. Where appropriate, report to 4 significant figures.

(a) Draw a Venn Diagram of the sample space for the type and location of an adsorbed molecule.
(b) What is the probability that a molecule is methane, given that it is adsorbed in zeolite-X?
(c) What is the probability that a molecule adsorbed in zeolite-Y?
(d) What is the probability that a molecule is ethane and in zeolite Y?
(e) What is the probability that a molecule is methane?

Solution:

We are given:

$$P(X) = 0.7$$
$$P(M \cap X) = 0.08$$
$$P(E \mid Y) = 0.8537$$

(a) Draw a Venn Diagram of the sample space for the type and location of an adsorbed molecule.

$M \cap X$	$M \cap Y$
$E \cap X$	$E \cap Y$

(b) What is the probability that a molecule is methane, given that it is adsorbed in zeolite-X?

$$P(M \mid X) = \frac{P(M \cap X)}{P(X)} = \frac{0.08}{0.7} = 0.1143$$

(c) What is the probability that a molecule adsorbed in zeolite-Y?

$$P(X) + P(Y) = 1$$

$$P(Y) = 1 - P(X) = 1 - 0.7 = 0.3$$

(d) What is the probability that a molecule is ethane and in zeolite Y?

$$P(E \cap Y) = P(E \mid Y)P(Y) = 0.8537 \cdot 0.3 = 0.2561$$

(e) What is the probability that a molecule is methane?

$$P(M) = P(M \cap X) + P(M \cap Y)$$

$$P(M \cap Y) + P(E \cap Y) = P(Y)$$

$$P(M \cap Y) = P(Y) - P(E \cap Y) = 0.3 - 0.2561 = 0.0439$$

$$P(M) = 0.08 + 0.0439 = 0.1239$$