Exam III Solutions
Administered: Wednesday, November 5, 2025
20 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (10 points)

In the elastic region, the stress-strain relationship is governed by Hooke's law in which the stress, σ, is related to the strain, ϵ, by elastic constants, c. Using Voigt notation, we can express this in three dimensions as

$$\underline{\sigma} = \underline{c\epsilon}$$

where

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \qquad \underline{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

For a material with a cubic crystal structure, there are only three independent non-zero elastic constants, C_{11}, C_{12} and C_{44}, leading to an elastic constant matrix

$$\underline{c} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

You are working with aluminum (FCC), which has values of these elastics given by

$$C_{11} = 104\ GPa \qquad C_{12} = 51\ GPa \qquad C_{44} = 159\ GPa$$

You perform an experiment where you measure the stresses in the materials to be

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0.1907 \\ 0.1854 \\ 0.1801 \\ 0.0159 \\ 0.0477 \\ 0.0795 \end{bmatrix}$$

(a) Are these equations linear or non-linear?
(b) Rearrange the system of equations into the form of $\underline{A}x = \underline{b}$. Identify each variable.
(c) Provide the determinant of the matrix of elastic constants.
(d) How many solutions are there to this system of equations?
(e) Determine the full set of strains within the material, $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}$ and ϵ_{23}.

Solution:

(a) Are these equations linear or non-linear?

The equations are linear. The unknowns are multiplied by constants and added together.

(b) Rearrange the system of equations into the form of $\underline{A}x = \underline{b}$. Identify each variable.

We can rearrange Hooke's law

$$\underline{\sigma} = \underline{c}\underline{\epsilon}$$

as

$$\underline{c}\underline{\epsilon} = \underline{\sigma}$$

Comparing this form to

$$\underline{A}\underline{x} = \underline{b}$$

where

$$\underline{A} = \underline{c}$$
$$\underline{x} = \underline{\epsilon}$$
$$\underline{b} = \underline{\sigma}$$

(c) Provide the determinant of the matrix of elastic constants.

Per the Python code provided below, the determinant of the matrix of elastic constants is

Determinant of A: 2326003332066.0

(d) How many solutions are there to this system of equations?

Because the determinant is not zero, there is one unique solution to the system of equations.

(e) Determine the full set of strains within the material, $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}$ and ϵ_{23}.

Per the Python code provided below, the vector

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.0009 \\ 0.0008 \\ 0.0001 \\ 0.0003 \\ 0.0005 \end{bmatrix}$$

Accounting for the factor of two in the off-diagonal components, provides all six elements of the strain matrix

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} 0.001 \\ 0.0009 \\ 0.0008 \\ 0.00005 \\ 0.00015 \\ 0.00025 \end{bmatrix}$$

*** python script start ***

```
import numpy as np

# Define stiffness matrix

n = 6
C = np.zeros((n,n))

C11 = 104.0
C12 = 51.0
C44 = 159.0

C[0,0] = C11
C[1,1] = C11
C[2,2] = C11
C[0,1] = C12
C[0,2] = C12
C[1,0] = C12
C[1,2] = C12
C[2,0] = C12
C[2,1] = C12
C[3,3] = C44
C[4,4] = C44
C[5,5] = C44

# Define stress vector

sig = np.zeros(n)
sig = np.array([0.1907, 0.1854, 0.1801, 0.0159, 0.0477, 0.0795])

# Compute determinant of C
detC = np.linalg.det(C)
print(f"Determinant of A: {detC}")

# Compute inverse of C
invC = np.linalg.inv(C)
# print(f"(f) Inverse of C:\n{invC}")

# Solve Ceps = sig
eps = np.dot(invC, sig)
print(f"Solution to C*eps = sig: {eps}")

eps11 = eps[0]
eps22 = eps[1]
eps33 = eps[2]
```

```
eps23 = 0.5*eps[3]
eps13 = 0.5*eps[4]
eps12 = 0.5*eps[5]
print(f" eps11 = {eps11}")
print(f" eps22 = {eps22}")
print(f" eps33 = {eps33}")
print(f" eps23 = {eps23}")
print(f" eps13 = {eps13}")
print(f" eps12 = {eps12}")
```

*** python script end ***

When run, this script yields the following output:

```
Determinant of A: 2326003332065.999
Solution to C*eps = sig: [0.001  0.0009 0.0008 0.0001 0.0003 0.0005]
 eps11 = 0.0009999999999999998
 eps22 = 0.0009000000000000003
 eps33 = 0.0008000000000000001
 eps23 = 5e-05
 eps13 = 0.00015000000000000001
 eps12 = 0.00025
```

Problem 2. (10 points)
The file, http://utkstair.org/clausius/docs/mse301/data/xm3p02_f25.txt, contains results from a tensile test experiment. The tensile strain is given in column 1 and the tensile stress is given in column 2 with units of MPa.

(a) Plot the stress vs strain curve.
(b) Identify the region in which the elastic modulus can be extracted.
(c) Perform the linear regression and report the mean value of the elastic modulus.
(d) Also report the standard deviation of the elastic modulus.
(e) Report the measure of fit of the model.

Solution

I wrote a Python script to perform these four tasks. The script calls the linreg1.py function to perform the regression.

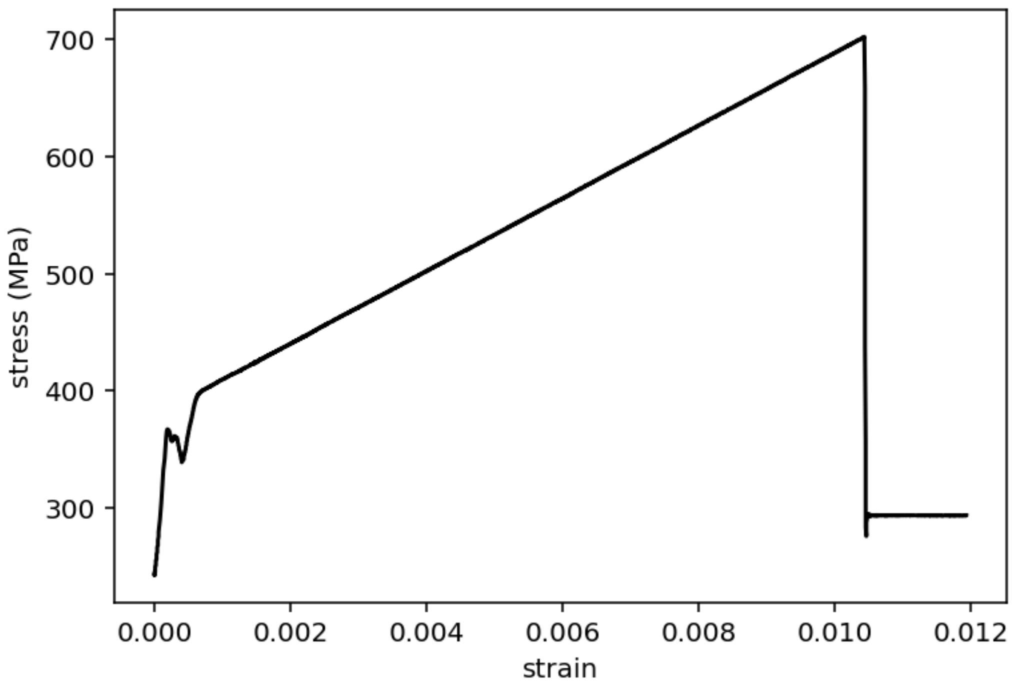
*** python script start ***

```
import numpy as np
import matplotlib.pyplot as plt
from linreg1 import linreg1

# read data
datamat = np.loadtxt('xm3p02_f25.txt')
ndata= datamat.shape[0]
print(f"There are {ndata} data points.")
strain = datamat[:,0]
stress = datamat[:,1]

# plot data
plt.plot(strain, stress, 'k-')
plt.xlabel('strain')
plt.ylabel('stress (MPa)')

# set limits of regression
# indexreglo = 0
# indexreghi = ndata - 1

indexreglo = 612
indexreghi = 3051

# perform regression
(b0, b1), (b0_sd, b1_sd), MOF = linreg1(strain[indexreglo:indexreghi+1],
                                        stress[indexreglo:indexreghi+1])

modulusav = b1
print(f"elastic modulus average = {modulusav}")
modulussd = b1_sd
print(f"elastic modulus standard deviation = {modulussd}")
print(f"MOF = {MOF}")
```

*** python script end ***

(a) Plot the stress vs strain curve.

The plot of the entire data is given below.



(b) Identify the region in which the elastic modulus can be extracted.

It looks like a region from strain of 0.002 to 0.10 is in the linear regime.
I went through the data file and found that this corresponds to indices in the strain vector from about 612 to 3051. These two numbers are entered in the code as the lower and upper limits of the regression.

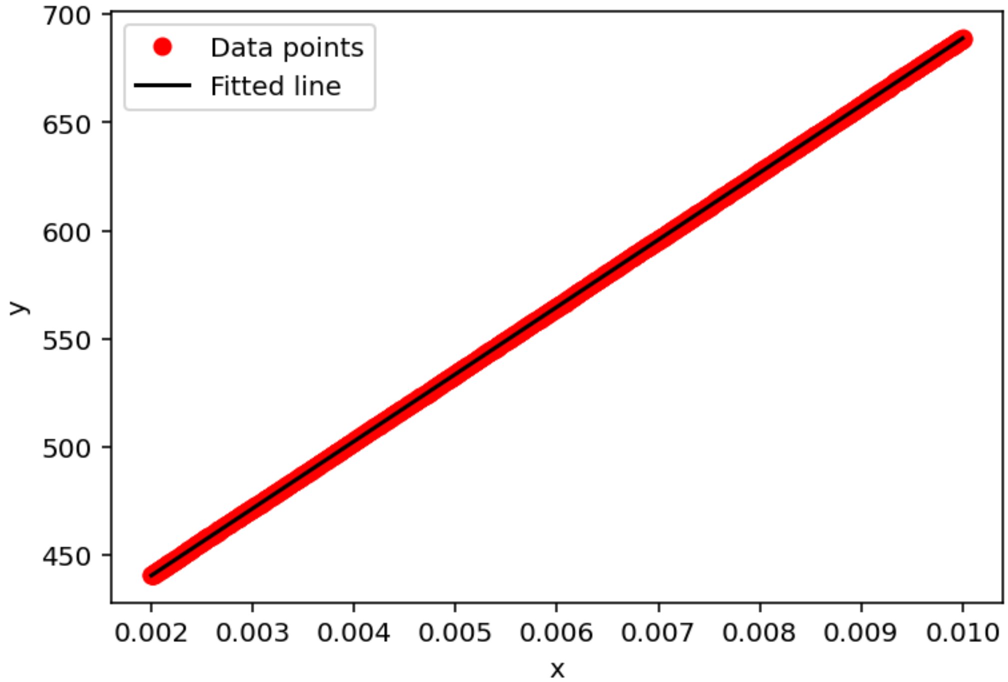
(c) Perform the linear regression and report the mean value of the elastic modulus.
(d) Also report the standard deviation of the elastic modulus.
(e) Report the measure of fit of the model.

Running the code generates the following output.

```
There are 3641 data points.
elastic modulus average = 31071.02183866715
elastic modulus standard deviation = 1.3367755838537543
MOF = 0.9999954872873732
```

The elastic modulus is 31,071 MPa or 31 GPa with a standard deviation of 1.3 MPa.
The measure of fit is 0.999995.

The linreg1.py function generates a second plot with the regression model fit to the segment of data used in the regression. This plot shows that in a region from strain of 0.002 to 0.10, the experimental data very closely follows a linear behavior.



An aside: (not required for the exam)

If you perform the regression on all of the data rather than the linear subset by changing two lines in the code from

```
indexreglo = 612
indexreghi = 3051
```

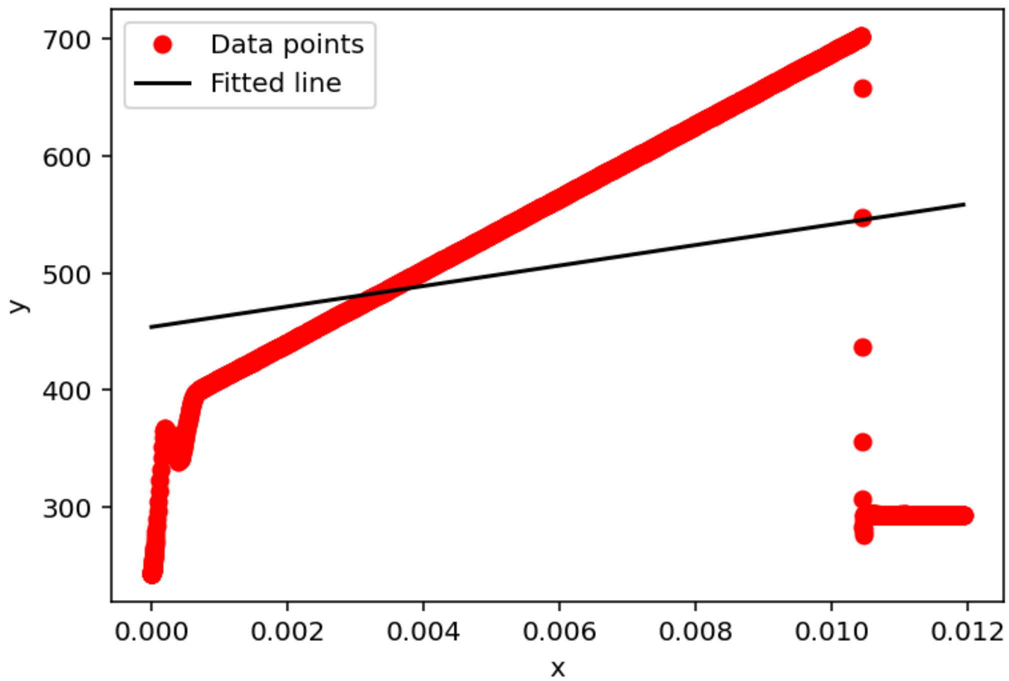
to

```
indexreglo = 0
indexreghi = ndata - 1
```

then the following output is generated

```
There are 3641 data points.
elastic modulus average = 8762.745719337456
elastic modulus standard deviation = 578.256859475137
MOF = 0.05935824214700352
```

with the corresponding plot



Clearly, a meaningful elastic modulus can only be obtained by regressing to the linear portion of the experimental data.