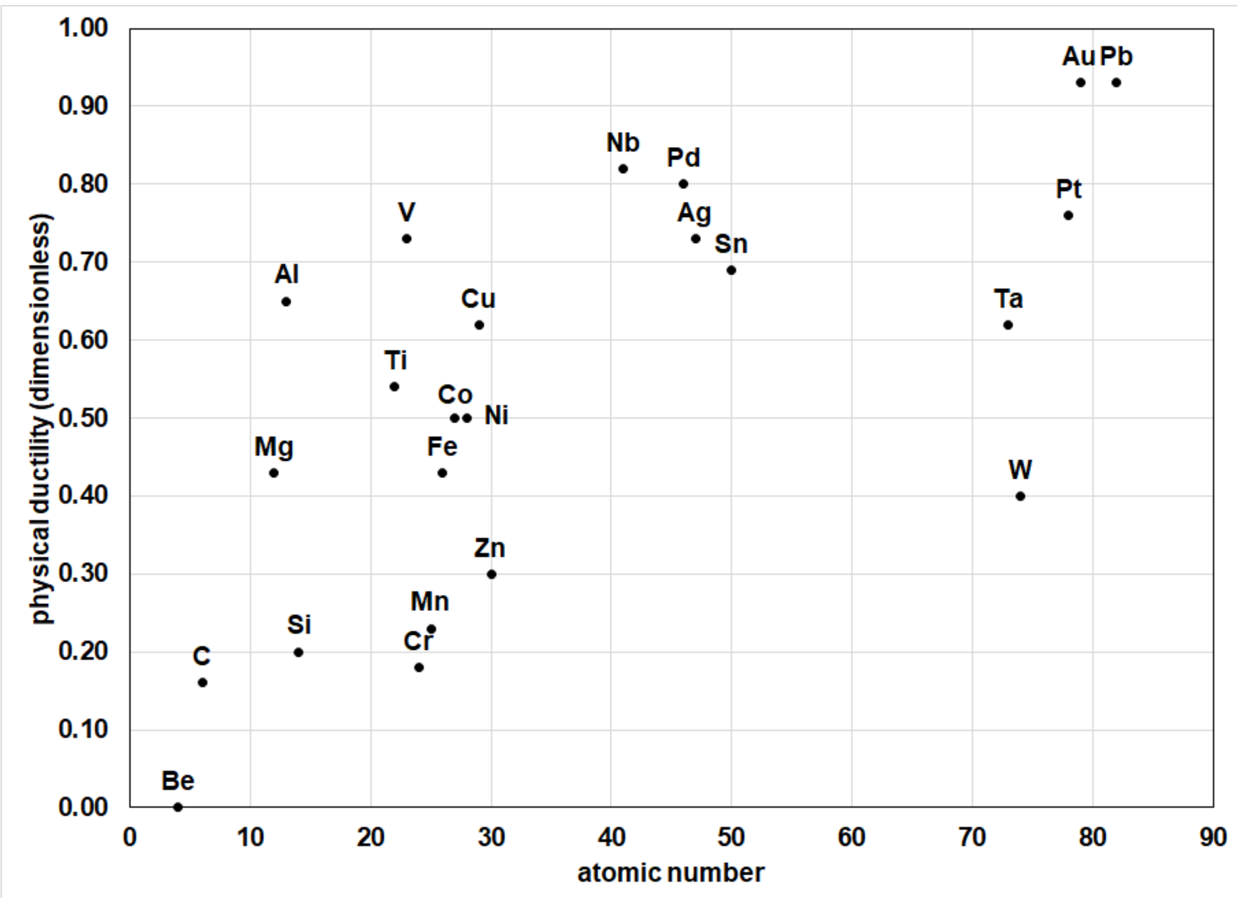
Exam I Solutions
Administered: Monday, September 15, 2025
24 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (12 points)

The "physical ductility" is a dimensionless parameter that ranges from 1 (perfectly ductile) to 0 (completely brittle). Consider the data for the following **23** materials plotted below. The data in this plot is available electronically on the course website in a spreadsheet file.



Answer the following questions for the materials in this table.

(a) Determine the mean atomic number.
(b) Determine the mean physical ductility.
(c) Determine the standard deviation of the atomic number.
(d) Determine the standard deviation of the physical ductility.
(e) Determine the correlation coefficient between the physical ductility and the atomic number.
(f) What is the physical significance of your answer to part (e)?

*Physical ductility data taken from https://www.failurecriteria.com/physicalductilit.html.

Solution:
(a) Determine the mean atomic number.

$$\mu_{AN} = \frac{\sum_{i=1}^{n} AN_i}{n} = 37.09$$

The mean atomic number of these 23 materials is 37.09.

(b) Determine the mean physical ductility.

$$\mu_{PD} = \frac{\sum_{i=1}^{n} PD_i}{n} = 0.528$$

The mean physical ductility of these 23 materials is 0.528.

(c) Determine the standard deviation of the atomic number.

$$\sigma_{\rho}^2 = E[AN^2] - E[AN]^2 = 1959.3 - (37.09)^2 = 583.9$$
$$\sigma_{AN} = \sqrt{\sigma_{AN}^2} = 24.16$$

The standard deviation of the atomic number is 24.16

(d) Determine the standard deviation of the physical ductility.

$$\sigma_{PD}^2 = E[PD^2] - E[PD]^2 = 0.3441 - (0.528)^2 = 0.0651$$

$$\sigma_{PD} = \sqrt{\sigma_{PD}^2} = 0.2551$$

The standard deviation of the physical ductility is 0.2551.

(e) Determine the correlation coefficient between the physical ductility and the atomic number.

First we determine the covariance.

$$\sigma_{PD\cdot AN} = E[PD \cdot AN] - E[PD]E[AN] = 23.69 - 0.528 \cdot 37.09 = 4.098$$

With the covariance in hand, we can determine the correlation coefficient.

$$\rho_{PD\cdot AN} = \frac{\sigma_{PD\cdot AN}}{\sigma_{PD} \cdot \sigma_{AN}} = \frac{4.098}{0.2551 \cdot 24.16} = 0.665$$

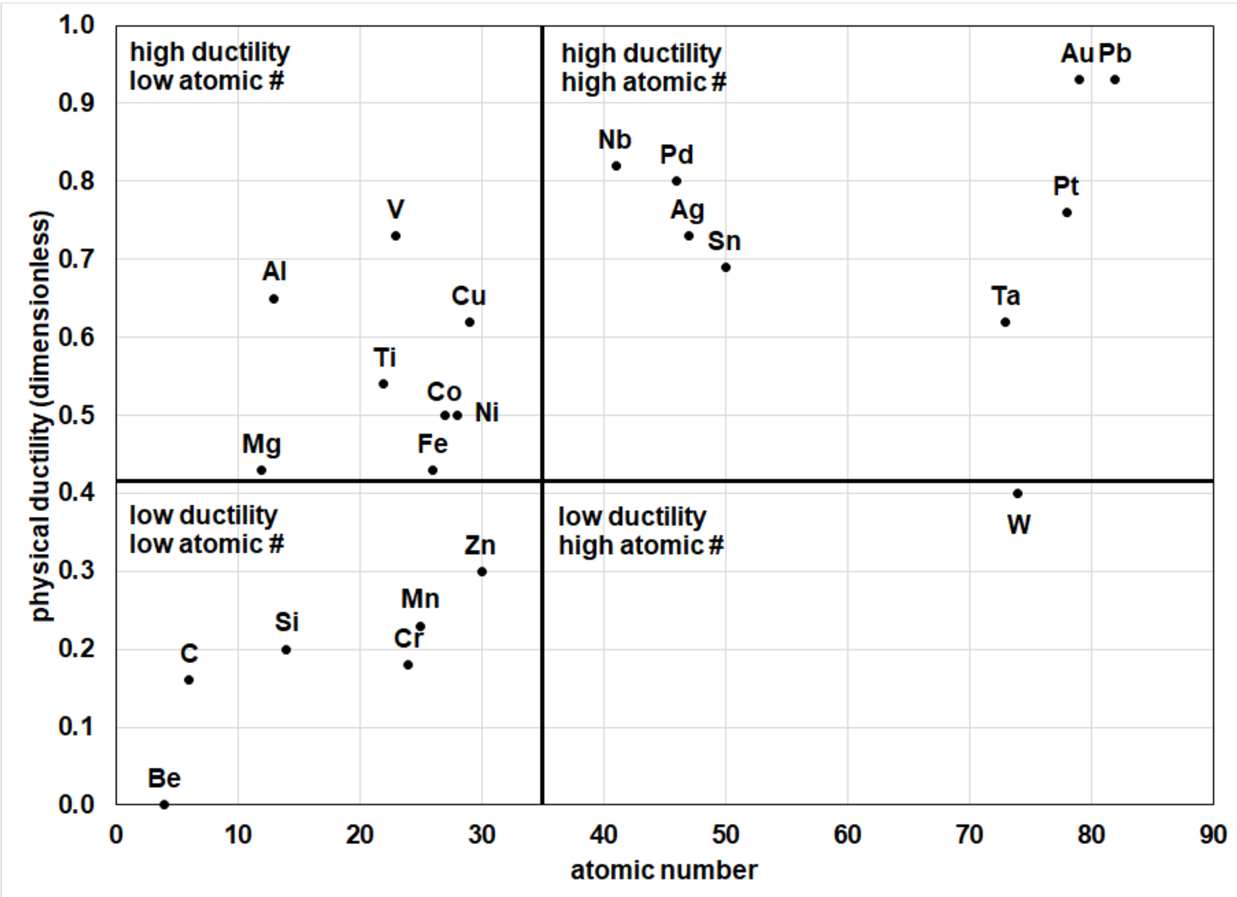
The correlation coefficient between the physical ductility and the atomic number is 0.67.

(f) What is the physical significance of your answer to part (e)?

The value of the correlation coefficient is positive. Therefore, there is a positive correlation between physical ductility and atomic number. An increase in the atomic number statistically corresponds to an increase in physical ductility. In class we said that in general a reliably strong correlation should have a correlation coefficient greater than 0.8. This correlation coefficient of 0.73 is not in that range. It is a positive correlation but not a strong one. You can see the extent of the deviation from the correlation in the plot.

Problem 2. (12 points)

Consider the **23** materials in the plot below. We are evaluating these materials in terms of physical ductility and atomic number. We have created a definition of low and high physical ductility and low and high atomic number as indicated by the horizontal and vertical lines in the plot.



Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this data.
(b) What is the probability that a material has high atomic number and high physical ductility?
(c) What is the probability that a material has high atomic number?
(d) What is the probability that a material has high physical ductility given that it has high atomic number?
(e) What is the probability that a material has high atomic number given that it has high physical ductility?
(f) Given this classification, prove that physical ductility and atomic number are not independent of each other.

Solution:
There are a total of 23 materials.
From the plot we see:

$$P(AN_{lo} \cap PD_{lo}) = \frac{6}{23}$$
$$P(AN_{hi} \cap PD_{lo}) = \frac{1}{23}$$
$$P(AN_{lo} \cap PD_{hi}) = \frac{8}{23}$$
$$P(AN_{hi} \cap PD_{hi}) = \frac{8}{23}$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$AN_{lo} \cap PD_{hi}$	$AN_{hi} \cap PD_{hi}$
$AN_{lo} \cap PD_{lo}$	$AN_{hi} \cap PD_{lo}$

(b) What is the probability that a material has high atomic number and high physical ductility?

From the data given,

$$P(AN_{hi} \cap PD_{hi}) = \frac{8}{23}$$

The probability that a material has high atomic number and high physical ductility is $\frac{8}{23}$.

(c) What is the probability that a material has high atomic number?

Consider the union probability rule.

$$P(AN_{hi}) = P(AN_{hi} \cap PD_{lo}) + P(AN_{hi} \cap PD_{hi}) - P[(AN_{hi} \cap PD_{lo}) \cap (AN_{hi} \cap PD_{hi})]$$

$$= \frac{1}{23} + \frac{8}{23} - 0 = \frac{9}{23}$$

The probability that a material has high atomic number and high physical ductility can be computed from the union rule. The intersection of low ductility and high ductility is zero by definition of the categorization of these materials. The probability that a material has a high atomic number is $\frac{9}{23}$.

(d) What is the probability that a material has high physical ductility given that it has high atomic number?

Consider the conditional probability rule.

$$P(PD_{hi}|AN_{hi}) = \frac{P(AN_{hi} \cap PD_{hi})}{P(AN_{hi})} = \frac{8}{23}\bigg/\frac{9}{23} = \frac{8}{9}$$

The probability that a material has high physical ductility given that it has high atomic number is $\frac{8}{9}$.

(e) What is the probability that a material has high atomic number given that it has high physical ductility?

There are two equivalent ways to work this problem.
First, consider the conditional probability rule.

$$P(AN_{hi}|PD_{hi}) = \frac{P(AN_{hi} \cap PD_{hi})}{P(PD_{hi})}$$

We can determine the denominator as

$$P(PD_{hi}) = P(AN_{lo} \cap PD_{hi}) + P(AN_{hi} \cap PD_{hi}) - P[(AN_{lo} \cap PD_{hi}) \cap (AN_{hi} \cap PD_{hi})]$$

$$= \frac{8}{23} + \frac{8}{23} - 0 = \frac{16}{23}$$

$$P(AN_{hi}|PD_{hi}) = \frac{P(AN_{hi} \cap PD_{hi})}{P(PD_{hi})} = \frac{8}{23}\Big/\frac{16}{23} = \frac{8}{16} = \frac{1}{2}$$

The probability that a material has high specific gravity given that it has low tensile strength is $\frac{1}{2}$.

A second, alternate method to reach the same answer is to write the two conditional probabilities below

$$P(PD_{hi}|AN_{hi}) = \frac{P(AN_{hi} \cap PD_{hi})}{P(AN_{hi})}$$

$$P(AN_{hi}|PD_{hi}) = \frac{P(AN_{hi} \cap PD_{hi})}{P(PD_{hi})}$$

and solve for the intersection

$$P(AN_{hi} \cap PD_{hi}) = P(PD_{hi}|AN_{hi})P(AN_{hi}) = P(AN_{hi}|PD_{hi})P(PD_{hi})$$

Solving the second equality for$P(AN_{hi}|PD_{hi})$ yields

$$P(AN_{hi}|PD_{hi}) = \frac{P(PD_{hi}|AN_{hi})P(AN_{hi})}{P(PD_{hi})} = \frac{\frac{8}{9}\frac{9}{23}}{\frac{16}{23}} = \frac{8}{16} = \frac{1}{2}$$

(f) Given this classification, prove that physical ductility and atomic number are not independent of each other.

There are a variety of ways to prove this.

If A & B are independent, then $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

So for example, we can show that hardness and density are not independent of each other through either of the following statements.

$$P(AN_{hi}|PD_{hi}) = \frac{1}{2} \neq P(AN_{hi}) = \frac{9}{23}$$

$$P(PD_{hi}|AN_{hi}) = \frac{8}{9} \neq P(PD_{hi}) = \frac{16}{23}$$

Since these statements are not equalities, we know that the variables are not independent of each other.