Exam I Solutions
Administered: Wednesday, September 20, 2023
24 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, <u>if work is shown</u>,
2 points for correct numerical value of solution

Problem 1. (12 points)

Consider the data for the following **17** refractory ceramics given below. This data is available electronically on the course website in a spreadsheet file.

Material	Melting Point	Max Temp	Hardness	Density	Specific Heat	Thermal Expansion (Linear)	Thermal Conductivity
	°C	°C	Moh's Scale	g/cm3	J/kg °C	10-6 / °C	W/m °C
Alumina	2050	1950	9	3.96	1050	8	4
Beryllia	2550	2400	9	3	2180	7.5	29
Magnesia	2850	2400	6	3.6	1170	13.5	59
Thoria	3220	2700	7	9.7	290	9.5	3
Zirconia	2700	2400	6.5	5.6	590	7.5	3
Zircon	2500	1870	7.5	4.6	630	4.5	4
Spinel	2130	1900	8	3.6	1050	8.5	2
Mullite	1850	1800	8	2.8	840	5	4
Sillimanite	1800	1800	6.5	3.2	840	5	2
Silicon Carbide	2200	1400	9	3.2	840	4.5	13
Silicon Nitride	1900	1400	9	3.18	1050	2.9	9.5
Graphite	3600	3273	0.75	2.2	1600	2.2	147
Quartizite	1400	3000	7	2.65	1170	8.6	2.6
Boron Carbide	2350	540	9.3	2.5	2090	5.7	17.3
Boron Nitride	2721	650	2	2.1	1570	7.5	26
Titanium Carbide	3140	1500	9.5	6.5	1050	6.9	40
Tungsten Carbide	2780	1000	9.5	14.3	300	6.3	43.3

Answer the following questions for the materials in this table.

(a) Determine the mean density.
(b) Determine the mean hardness.
(c) Determine the standard deviation of the density.
(d) Determine the standard deviation of the hardness.
(e) Determine the correlation coefficient between the density and the hardness.
(f) What is the physical significance of your answer to part (e)?

Solution:
(a) Determine the mean density.

$$\mu_\rho = \frac{\sum_{i=1}^{n} \rho_i}{n} = 4.511176471 \ {}^{g}/_{cm^3}$$

The mean density of these 17 materials is $4.51 \ {}^{g}/_{cm^3}$.

(b) Determine the mean hardness.

$$\mu_h = \frac{\sum_{i=1}^{n} h_i}{n} = 7.267647059 \ Mohs$$

The mean hardness of these 17 materials is 7.3 $Mohs$.
(c) Determine the standard deviation of the density.

$$\sigma_\rho^2 = E[\rho^2] - E[\rho]^2 = 29.70038235 - (4.511176471)^2 = 9.349669204 \left({}^{g}/_{cm^3}\right)^2$$

$$\sigma_\rho = \sqrt{\sigma_\rho^2} = 3.057722879\ {}^{g}/_{cm^3}$$

The standard deviation of the density is 3.06 ${}^{g}/_{cm^3}$.

(d) Determine the standard deviation of the hardness.

$$\sigma_h^2 = E[h^2] - E[h]^2 = 58.72367647 - (7.267647059)^2 = 5.904982699(Mohs)^2$$

$$\sigma_{C_p} = \sqrt{\sigma_{C_p}^2} = 2.430017016\ Mohs$$

The standard deviation of the hardness is 2.4 $Mohs$.

(e) Determine the correlation coefficient between the density and the hardness.

First we determine the covariance.

$$\sigma_{\rho\cdot h} = E[\rho\cdot h] - E[\rho]E[h] = 35.15941176 - 4.511176471 \cdot 7.267647059 = 2.373773356\ {}^{g}/_{cm^3}\ \text{Mohs}$$

With the covariance in hand, we can determine the correlation coefficient.

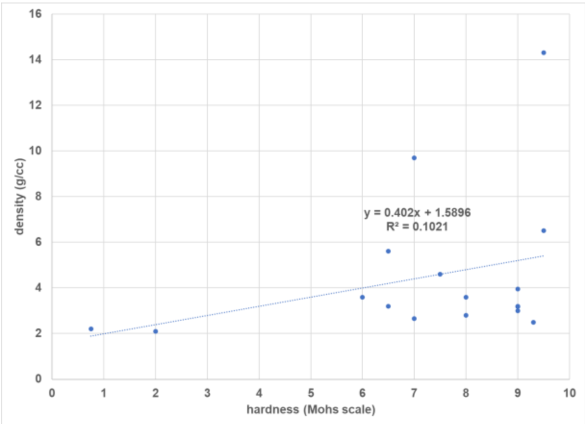
$$\rho_{\rho\cdot h} = \frac{\sigma_{\rho\cdot h}}{\sigma_\rho \cdot \sigma_h} = \frac{2.373773356}{3.057722879\cdot 2.430017016} = 0.319471274$$

The correlation coefficient of density and hardness is 0.32

(f) What is the physical significance of your answer to part (e)?

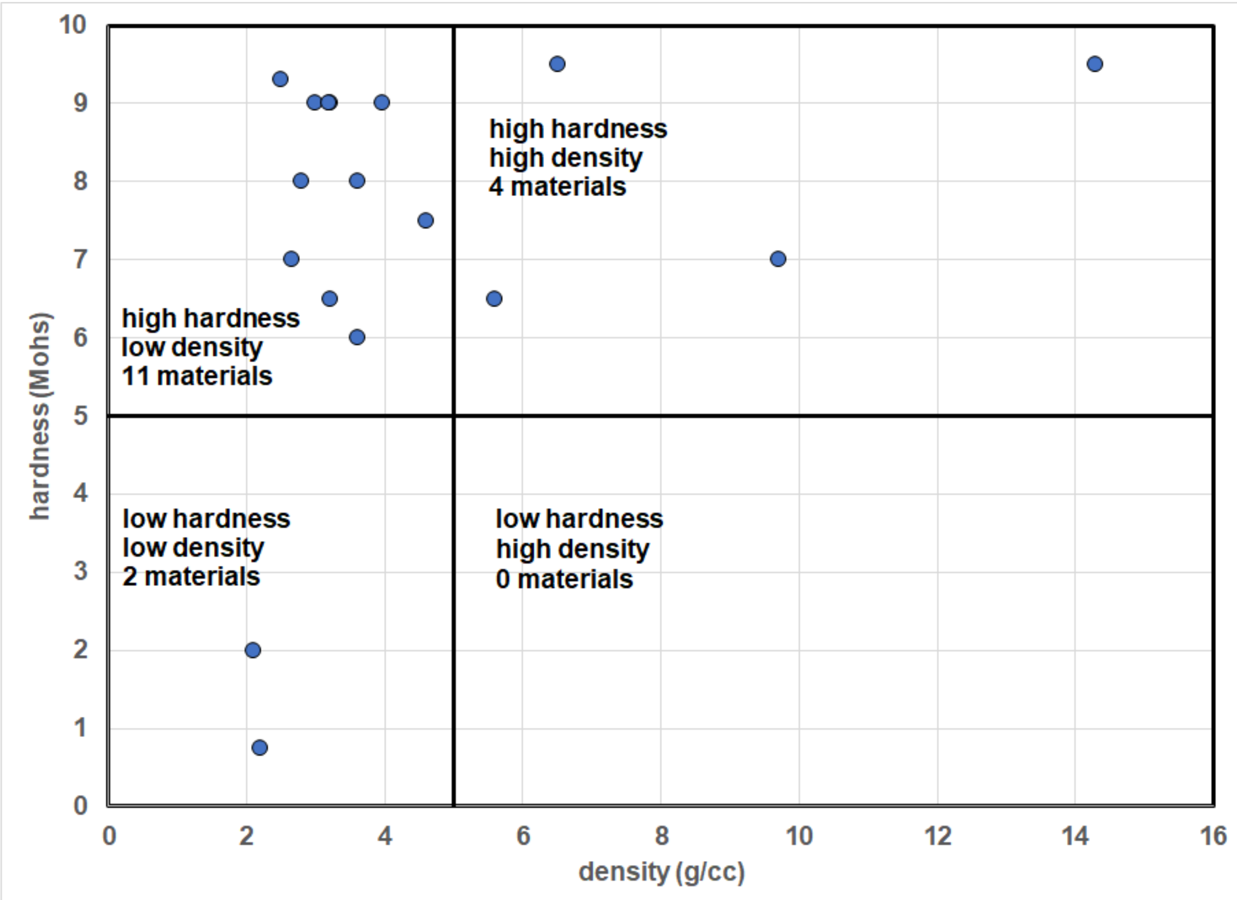
The value of the correlation coefficient is positive but on the scale from 0 to 1, closer to 0 than 1. Therefore, there is a weak positive correlation between density and hardness. An increase in the density statistically correspond to a modest increase in the hardness.

A plot of the density vs the hardness is shown below (with a linear regression). Plot not required for solution.



Problem 2. (12 points)

Consider the **17** ceramic materials in the table in Problem 1. We are evaluating these materials in terms of low or high hardness and low or high density. A plot of the hardness vs the density is shown below.



Using this information, answer the following questions.

(a) Draw a Venn Diagram of the sample space for this data.
(b) What is the probability that a material has high hardness and low density?
(c) What is the probability that a material has high hardness?
(d) What is the probability that a material has low density given that it has high hardness?
(e) What is the probability that a material has high hardness given that it had low density?
(f) Given this classification, prove that hardness and density are not independent of each other.

Solution:

There are a total of 17 materials.
From the plot we see:

$$P(h_{lo} \cap \rho_{lo}) = \frac{2}{17}$$
$$P(h_{hi} \cap \rho_{lo}) = \frac{11}{17}$$
$$P(h_{lo} \cap \rho_{hi}) = \frac{0}{17}$$
$$P(h_{hi} \cap \rho_{hi}) = \frac{4}{17}$$

(a) Draw a Venn Diagram of the sample space for this experiment.

$h_{hi} \cap \rho_{lo}$	$h_{lo} \cap \rho_{hi}$
$h_{lo} \cap \rho_{lo}$	$h_{lo} \cap \rho_{hi}$

(b) What is the probability that a material has high hardness and low density?

From the data given,

$$P(h_{hi} \cap \rho_{lo}) = \frac{11}{17}$$

The probability that a material has high hardness and low density is $\frac{11}{17}$.

(c) What is the probability that a material has high hardness?

Consider the union probability rule.

$$P(h_{hi}) = P(h_{hi} \cap \rho_{hi}) + P(h_{hi} \cap \rho_{lo}) - P[(h_{hi} \cap \rho_{lo}) \cap (h_{hi} \cap \rho_{hi})]$$
$$= \frac{4}{17} + \frac{11}{17} - 0 = \frac{15}{17}$$

The probability that a material has a high hardness can be computed from the union rule. The intersection of low density and high density is zero by definition of the categorization of these materials. The probability that a material has a high hardness is $\frac{15}{17}$.

(d) What is the probability that a material has low density given that it has high hardness?

Consider the conditional probability rule.

$$P(\rho_{lo}|h_{hi}) = \frac{P(h_{hi} \cap \rho_{lo})}{P(h_{hi})} = \frac{11}{17}\bigg/\frac{15}{17} = \frac{11}{15}$$

The probability that a material has low density given that it has high hardness is $\frac{11}{15}$. More than two thirds of the materials in this data set with high hardness have low density.

(e) What is the probability that a material has high hardness given that it had low density?

Consider the conditional probability rule.

$$P(h_{hi}|\rho_{lo}) = \frac{P(h_{hi} \cap \rho_{lo})}{P(\rho_{lo})}$$

We can determine the denominator as

$$P(\rho_{lo}) = P(h_{hi} \cap \rho_{lo}) + P(h_{lo} \cap \rho_{lo}) - P[(h_{hi} \cap \rho_{lo}) \cap (h_{lo} \cap \rho_{lo})]$$
$$= \frac{11}{17} + \frac{2}{17} - 0 = \frac{13}{17}$$

$$P(h_{hi}|\rho_{lo}) = \frac{P(h_{hi} \cap \rho_{lo})}{P(\rho_{lo})} = \frac{11}{17}\bigg/\frac{13}{17} = \frac{11}{13}$$

The probability that a material has high hardness given that it has low density is $\frac{11}{13}$. Most materials in this data set with low density have high hardness.

(f) Given this classification, prove that hardness and density are not independent of each other.

There are a variety of ways to prove this.

If A & B are independent, then $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

So for example, we can show that hardness and density are not independent of each other through either of the following statements.

$$P(\rho_{lo}|h_{hi}) = \frac{11}{15} \neq P(\rho_{lo}) = \frac{13}{17}$$

$$P(h_{hi}|\rho_{lo}) = \frac{11}{13} \neq P(h_{hi}) = \frac{15}{17}$$

Since these statements are not equalities, we know that the variables are not independent of each other.