Exam II
Administered: Wednesday, October 8, 2003
28 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

Problem 1. (10 points)

We are studying two different methods to determine diffusion coefficients from molecular simulation. In the first method, using the Darken equation, we run 10 simulations and find a sample mean value of the diffusivity to be 5.0x10^{-8} m^2/s with a sample variance of 0.04x10^{-16} m^4/s^2. In the second method, using Linear Irreversible Thermodynamics (LIT), we run 8 simulations and find a sample mean value of the diffusivity to be 6.2x10^{-8} m^2/s with a sample variance of 25.0x10^{-16} m^4/s^2.

Based on this information, answer the following questions.

(a) What PDF is appropriate for determining a confidence interval on the difference of means?
(b) Find the lower limit on a 95% confidence interval on the difference of means.
(c) Find the upper limit on a 95% confidence interval on the difference of means.
(d) Are we 95% confident that the Darken equation gives diffusivities within 2.0x10^{-8} m^2/s of the LIT method?
(e) Does this data support the claim that the Darken equation yields statistically the same results as LIT, for the given level of confidence?

Solution:

(a) What PDF is appropriate for determining a confidence interval on the difference of means?

We must use the t-distribution to determine the confidence interval on the difference of means when the population variances are unknown.

(b) Find the lower limit on a 95% confidence interval on the difference of means.
(c) Find the upper limit on a 95% confidence interval on the difference of means.

$$\bar{x}_1 = 5.0\cdot 10^{-8} \qquad \bar{x}_2 = 6.2\cdot 10^{-8} \qquad n_1 = 10 \qquad n_2 = 8$$

$$s_1^2 = 0.04\cdot 10^{-16} \qquad s_2^2 = 25.0\cdot 10^{-16}$$

$$s_1 = 0.2\cdot 10^{-8} \qquad s_2 = 5.0\cdot 10^{-8}$$

$$\alpha = \frac{1-\text{C.I.}}{2} = 0.025$$

$$v = \frac{\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}\right)^2}{\left[\left(\frac{s_1^2}{n_1}\right)^2 \Big/ (n_1-1)\right] + \left[\left(\frac{s_2^2}{n_2}\right)^2 \Big/ (n_2-1)\right]} \text{ if } \sigma_1 \neq \sigma_2$$

$$v = 7.0179 \approx 7$$

$$t_\alpha(v) = t_{0.025}(7) = 2.365 \text{ from Table A.4 of WMM}$$

$$t_{1-\alpha} = -t_\alpha$$

confidence interval:

$$P\left[\left(\overline{X}_1-\overline{X}_2\right)+t_{1-\alpha}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}<\left(\mu_1-\mu_2\right)<\left(\overline{X}_1-\overline{X}_2\right)+t_{\alpha}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}\right]=1-2\alpha$$

$$P\left[-5.383\cdot10^{-8}<\left(\mu_1-\mu_2\right)<2.983\cdot10^{-8}\right]=0.95$$

(d) Are we 95% confident that the Darken equation gives diffusivities within $2.0\text{x}10^{-8}$ m^2/s of the LIT method?

No, while the range of $\Delta\mu$ does include $-2.0\text{x}10^{-8}$ m^2/s to $+2.0\text{x}10^{-8}$ m^2/s, the actual range of the interval is larger than this.

If someone were to ask if it is possible that the real $\Delta\mu$ is between $-2.0\text{x}10^{-8}$ m^2/s to $+2.0\text{x}10^{-8}$ m^2/s, then we would have to say, "Yes, the data allows for that possibility."

(e) Does this data support the claim that the Darken equation yields statistically the same results as LIT, for the given level of confidence?

Because the confidence interval ranges from negative values where $\mu_2 > \mu_1$ to positive values, where $\mu_1 > \mu_2$, our data cannot invalidate the claim that the Darken equation yields statistically the same results as LIT.

Problem 2. (12 points)

A particular manufacturer makes tires for both automobiles and motorcycles with the same mean life time of 60,000 miles.

(a) What PDF would describe the probability that an individual tire is operating after 50,000 miles?
(b) What is the probability that an individual tire is operating after 50,000 miles?
(c) What PDF would describe the probability that all the tires on a car are still functioning after 50,000 miles?
(d) What is the probability that all the tires on a car are still functioning after 50,000 miles?
(e) What is the probability that all the tires on a motorcycle are still functioning after 50,000 miles?
(f) Explain why your answer to (d) is greater than (e) or why (e) is greater than (d), as the case may be?

Solution:

(a) What PDF would describe the probability that an individual tire is operating after 50,000 miles?

The exponential PDF.

(b) What is the probability that an individual tire is operating after 50,000 miles?

$$P(t>50k)=\int_{50k}^{\infty}f_e(t;\beta)dt=\int_{50k}^{\infty}\frac{1}{\beta}e^{-\frac{t}{\beta}}dt=-e^{-\frac{t}{\beta}}\Bigg|_{50k}^{\infty}=-e^{-\frac{\infty}{\beta}}--e^{-\frac{50k}{\beta}}=0+e^{-\frac{50k}{\beta}}$$

$$P(t>50k)=e^{-\frac{50000}{60000}}=0.4346$$

(c) What PDF would describe the probability that all the tires on a car are still functioning after 50,000 miles?

This calls for the binomial probability because the tires are independent and all have the same probability of operating after 50,000 miles.

(d) What is the probability that all the tires on a car are still functioning after 50,000 miles?
x = random variable = number of functioning tires = 4
n = total number of tires = 4
p = probability that an individual tire still functioning after 50000 miles = answer to part (b)

$$P(X = x) = b(x;n,p) = \binom{n}{x} p^x q^{n-x}$$

$$P(x = 4) = b(4;4,0.4346) = 0.03567$$

(e) What is the probability that all the tires on a motorcycle are still functioning after 50,000 miles?
x = random variable = number of functioning tires = 2
n = total number of tires = 2
p = probability that an individual tire still functioning after 50000 miles = answer to part (b)

$$P(x = 4) = b(2;2,0.4346) = 0.1889$$

(f) Explain why your answer to (d) is greater than (e) or why (e) is greater than (d), as the case may be?

Part (e) is greater than (d). It is more likely to have 2 good tires on a motorcycle than 4 good tires on a car because a car has more tires, thus giving it more opportunity to blow one out.

"Part (e) is greater than (d) because with 2 wheels, there are less opportunities for tire failure. It's the same as flipping coins; more trials cause all outcomes being the same to be less likely."—D.A. (A student's answer to this problem.)

Had we been asked which is more likely to have at least 2 good tires, then the answer would be the automobile. (p=0.5836 for car, but same answer as in part (e) for motorcycle.)

Problem 3. (6 points)

We run a warranty company that provides replacement parts for digital cameras. If our research team tells us that on average digital cameras have a lifetime of 4 years with a standard deviation of 2 years, then answer the following questions.

(a) If we provide a warranty for all cameras lasting less than 1 years, what fraction of the cameras can we expect to replace?

(b) If we only want to replace 5% of the cameras, how long should our warranty last?

(c) What PDF did you use to solve (a) & (b)?

Solution:

(a) If we provide a warranty for all cameras lasting less than 1 years, what fraction of the cameras can we expect to replace?

$$P(x < 1) = P(z < \frac{x-\mu}{\sigma}) = P(z < \frac{1-4}{2}) = P(z < -1.5) = 0.0668$$

From table A.3.
We can expect 6.68 percent of the cameras to fail before one year.

(b) If we only want to replace 5% of the cameras, how long should our warranty last?

$$P(z < z_{lo}) = 0.05$$

$$z_{lo} = -1.645 \qquad \text{from table A.3}$$

$$z_{lo} = \frac{x_{lo} - \mu}{\sigma}$$

$$x_{lo} = \mu + \sigma z_{lo} = 4 + 2(-1.645) = 0.71 \text{ years}$$

Our warranty should last0.71 years

(c) What PDF did you use to solve (a) & (b)?

Normal distribution. The variable is continuous and we have only been provided the mean and the standard deviation.