Derivation of the fact that the distribution of the sample mean is the normal distribution

Consider taking n samples from a population characterized by mean, μ, and variance, σ^2. The sample mean is given by $\bar{x}$.

We define a moment generating function for a continuous PDF to be:

$$M_x(t) = \mu_{e^{tx}} = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

From this definition, and using the rules of linear operation, we can show that

$$M_{x+a}(t) = \mu_{e^{t(x+a)}} = E\left[e^{t(x+a)}\right] = \int_{-\infty}^{\infty} e^{t(x+a)} f(x)dx = \int_{-\infty}^{\infty} e^{tx}e^{ta} f(x)dx = e^{ta}\int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$M_{x+a}(t) = e^{ta}M_x(t) \qquad (1)$$

$$M_{ax}(t) = \mu_{e^{tax}} = E\left[e^{tax}\right] = \int_{-\infty}^{\infty} e^{tax} f(x)dx = M_x(at)$$

$$M_{ax}(t) = M_x(at) \qquad (2)$$

$$M_{x_1+x_2+x_3+\ldots+x_n}(t) = \mu_{e^{t(x_1+x_2+x_3+\ldots+x_n)}} = E\left[e^{t(x_1+x_2+x_3+\ldots+x_n)}\right] = \int_{-\infty}^{\infty} e^{t(x_1+x_2+x_3+\ldots+x_n)} f(x)dx$$

$$M_{x_1+x_2+x_3+\ldots+x_n}(t) = M_{x_1}(t)M_{x_2}(t)M_{x_3}(t)\ldots M_{x_n}(t) \qquad (3)$$

So that

$$M_{(\bar{x}-\mu)/(\sigma/\sqrt{n})}(t) = \mu_{e^{t\left[(\bar{x}-\mu)/(\sigma/\sqrt{n})\right]}} = E\left[e^{t\left[(\bar{x}-\mu)/(\sigma/\sqrt{n})\right]}\right] = \int_{-\infty}^{\infty} e^{t\left[(\bar{x}-\mu)/(\sigma/\sqrt{n})\right]} f(\bar{x})d\bar{x}$$

$$M_{(\bar{x}-\mu)/(\sigma/\sqrt{n})}(t) = \int_{-\infty}^{\infty} e^{t\bar{x}/(\sigma/\sqrt{n})}e^{-t\mu/(\sigma/\sqrt{n})} f(\bar{x})d\bar{x} = e^{-t\mu\sqrt{n}/\sigma}\int_{-\infty}^{\infty} e^{t\bar{x}\sqrt{n}/\sigma} f(\bar{x})d\bar{x}$$

$$M_{(\bar{x}-\mu)/(\sigma/\sqrt{n})}(t) = e^{-t\mu\sqrt{n}/\sigma}M_{\bar{x}}\left(\frac{t\sqrt{n}}{\sigma}\right)$$

Now consider that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, so

$$M_{\bar{x}}\left(\frac{t\sqrt{n}}{\sigma}\right) = M_{\frac{1}{n}\sum_{i=1}^{n} X_i}\left(\frac{t\sqrt{n}}{\sigma}\right) = \int_{-\infty}^{\infty} e^{t\sqrt{n}/\sigma \frac{1}{n}\sum_{i=1}^{n} X_i} f(x)dx = \int_{-\infty}^{\infty} e^{t/(\sigma\sqrt{n})\sum_{i=1}^{n} X_i} f(x)dx$$

$$M_{\bar{x}}(t\sqrt{n}/\sigma) = \int_{-\infty}^{\infty} e^{t/(\sigma\sqrt{n})\sum_{i=1}^{n} X_i} f(x)dx = M_{x_1}\left(\frac{t}{\sigma\sqrt{n}}\right) M_{x_2}\left(\frac{t}{\sigma\sqrt{n}}\right) \ldots M_{x_n}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

Since there isn't anything intrinsic that distinguishes one X_i from another, we can write

$$M_{\bar{x}}(t\sqrt{n}/\sigma) = \left[M_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

If we substitute this back into our original equation

$$M_{\frac{(\bar{x}-\mu)}{(\sigma/\sqrt{n})}}(t) = e^{-t\mu\sqrt{n}/\sigma} M_{\bar{x}}\left(\frac{t\sqrt{n}}{\sigma}\right) = e^{-t\mu\sqrt{n}/\sigma}\left[M_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

Take the natural log of both sides:

$$\ln\left[M_{\frac{(\bar{x}-\mu)}{(\sigma/\sqrt{n})}}(t)\right] = \frac{-t\mu\sqrt{n}}{\sigma} + n\ln\left[M_x\left(\frac{t}{\sigma\sqrt{n}}\right)\right]$$

Expand $M_x\left(\frac{t}{\sigma\sqrt{n}}\right)$ as an infinite series in powers of t about t=0.

$$M_x\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + v_1 t + v_2\frac{t^2}{2!} + v_3\frac{t^3}{3!} + \ldots + v_r\frac{t^r}{r!} + \ldots$$

where

$$v_i = \left.\frac{d^i M_x\left(\frac{t}{\sigma\sqrt{n}}\right)}{dt^i}\right|_{t=0}$$

We can write this as

$$M_x\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + v(t)$$

where v(t) is an infinite series in t. For very large sample sizes, n

$$\lim_{n\to\infty} \ln\left[M_{\frac{(\bar{x}-\mu)}{(\sigma/\sqrt{n})}}(t) \right] = \lim_{n\to\infty} \ln[1+v(t)] = \frac{t^2}{2}$$

This can be shown by expanding the natural log in a Mclaurin series. For the present purposes, we will take this step given above on faith. Then, we have

$$\lim_{n\to\infty} M_{\frac{(\bar{x}-\mu)}{(\sigma/\sqrt{n})}}(t) = e^{\frac{t^2}{2}}$$

So the first moment of $Z = \dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ in the limit of large n is $e^{\frac{t^2}{2}}$

Well, let's find what the moment of the random variable, z, would be if it follows the normal distribution. The PDF of the normal distribution is

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$M_x(t) = \mu_{e^{tx}} = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2+2x\mu-\mu^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} e^{\frac{2tx\sigma^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2+2x\mu-\mu^2}{2\sigma^2}} dx$$

$$M_x(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{2tx\sigma^2-x^2+2x\mu-\mu^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2+2\left(t\sigma^2+\mu\right)x-\mu^2}{2\sigma^2}} dx$$

Complete the square in the exponent:

$$-x^2 + 2\left(t\sigma^2+\mu\right)x - \mu^2 = \left[x - \left(t\sigma^2+\mu\right)\right]^2 - 2\mu t\sigma^2 - t^2\sigma^4$$

$$M_x(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left[x-\left(t\sigma^2+\mu\right)\right]^2-2\mu t\sigma^2-t^2\sigma^4}{2\sigma^2}} dx = e^{\mu t+\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left[x-\left(t\sigma^2+\mu\right)\right]^2}{2\sigma^2}} dx$$

Let $w = \dfrac{\left[x-\left(t\sigma^2+\mu\right)\right]}{\sigma}$ so that $dw = \dfrac{dx}{\sigma}$ and

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw = e^{\mu t + \frac{t^2\sigma^2}{2}} (1) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

So that the first moment generating function of the standard normal PDF is

$$M_x(t) = e^{\frac{t^2}{2}}$$

If we compare this moment generating function with that obtained for

$$\lim_{n \to \infty} M_{\frac{(\bar{x}-\mu)}{(\sigma/\sqrt{n})}}(t) = e^{\frac{t^2}{2}}$$

we find that they are the same in the limit of large n. Since there is a one-to-one correspondence between PDFs and moment-generating functions, we see that the PDF for $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ is the standard normal PDF.