# Overview of the weighting constants and the points where we evaluate the function for The Gaussian quadrature

**Project two**

**By**
**Ashraf Marzouk**
**ChE 505**
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**Department of Mechanical Engineering**
**University of Tennessee**
**Knoxville, TN**

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## 1. Introduction

Gaussian quadrature seeks to obtain the best numerical estimate of an integral by picking optimal abscissas $x_i$ at which to evaluate the function f(x). The fundamental theorem of Gaussian quadrature states that the optimal abscissas of the m point Gaussian quadrature formulas are precisely the roots of the orthogonal polynomial for the same interval and weighting function. Gaussian quadrature is optimal because it fits all polynomials up to degree 2m exactly.

I will introduce Legendre polynomials and fitting a polynomial to a set of points because we will need them in the derivation of Gaussian quadrature.

### 1.1 Legendre Polynomials

Consider the expression

$$(1-2rx+r^2)^{-1/2} \qquad \rightarrow 1.1.1$$

in which |x| and |r| are both less than or equal to one. We can expand the expression 1.1.1 by the binomial theorem as a series of powers of r. This is straightforward, though not particularly easy, and we might expect to spend several minutes in obtaining the coefficients of the first few powers of r. We will find that the coefficient of $r^n$ is a polynomial expression in x of degree n. Indeed the expansion takes the form

$$(1-2rx+r^2)^{-1/2} = P_o(x) + P_1(x)\, r + P_2(x)\, r^2 + P_3(x)\, r^3 \ldots\ldots \rightarrow 1.1.2$$

The coefficients of the successive power of n are the Legendre polynomials; the coefficient of $r^n$, which is $P_n(x)$, is the Legendre polynomial of order n, and it is a polynomial in *x* including terms as high as $x_n$. If we have conscientiously tried to expand expression 1.1.1, you will have found that

$$P_o(x) = 1,\ P_1(x) = x,\ P_2(x) = 0.5(3x^2-1) \rightarrow 1.1.3$$

Though we probably gave up with exhaustion after that and didn't take it any further. If we look carefully at how we derived the first few polynomials, we may have discovered for ourselves that we can obtain the next polynomial as a function of two earlier polynomials. we may even have discovered for ourselves the following recursion relation:

$$P_{n+1} = \frac{(2n+1)xP_n - nP_{n-1}}{n+1} \rightarrow 1.1.4$$

This enables us very rapidly to obtain higher order Legendre polynomials, whether numerically or in algebraic form. For example, put n = 1 and show that equation 1.1.4 results in $P_2 = ½(3x^2 -1)$ we will then want to calculate $P_3$, and then $P_4$, and more and more and more.

Another way to generate them is form the equation

$$P_{n+1} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - n)^n \rightarrow 1.1.5$$

Here are the first eleven Legendre polynomials:

$$P_0 = 1$$
$$P_1 = x$$
$$P_2 = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3 = \tfrac{1}{2}(5x^3 - 3x)$$
$$P_4 = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5 = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$
$$P_6 = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$
$$P_7 = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$
$$P_8 = \tfrac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$$
$$P_9 = \tfrac{1}{128}(12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$$
$$P_{10} = \tfrac{1}{256}(46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$$

Inspection of the forms of these polynomials will quickly show that all odd polynomials have a root of zero, and all nonzero roots occur in positive/negative pairs. We have shall have no difficulty in finding the roots of these equations.
The roots $x_{n,i}$ , are in appendix A, which also lists certain coefficients $c_{n,i}$, that will be explained in section 1.2.

The graphs of the Legendre polynomials are in figures I and 2.

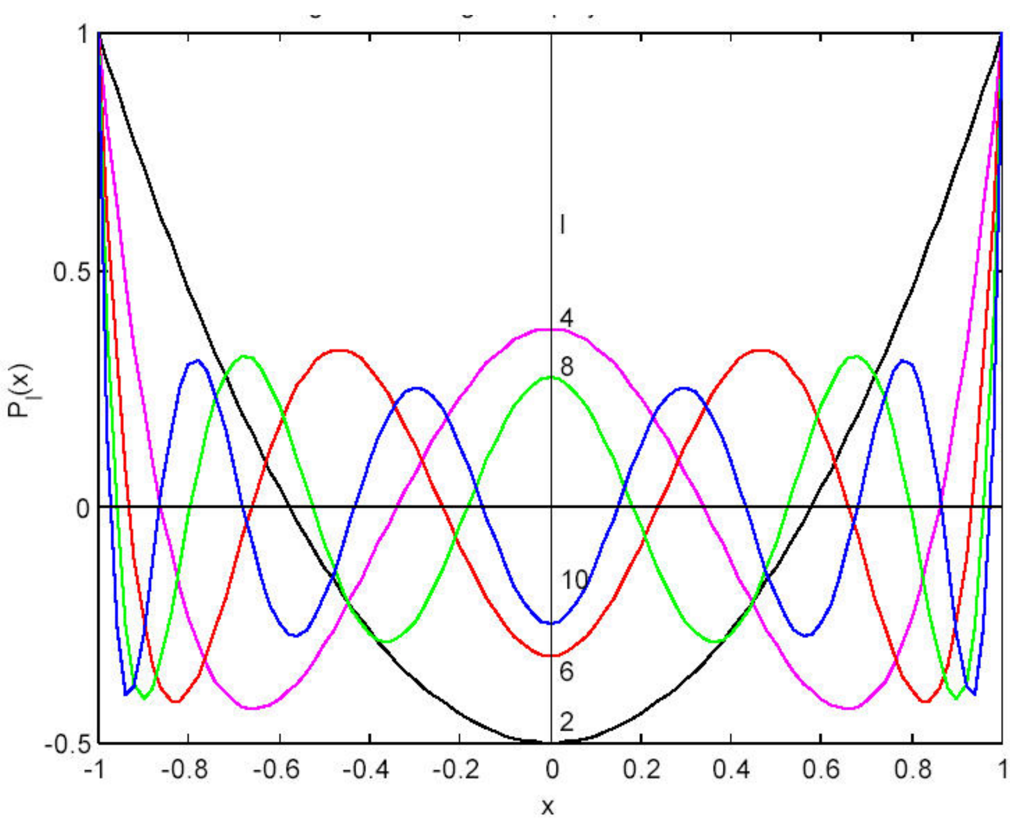


Figure I Legendre polynomials for even n

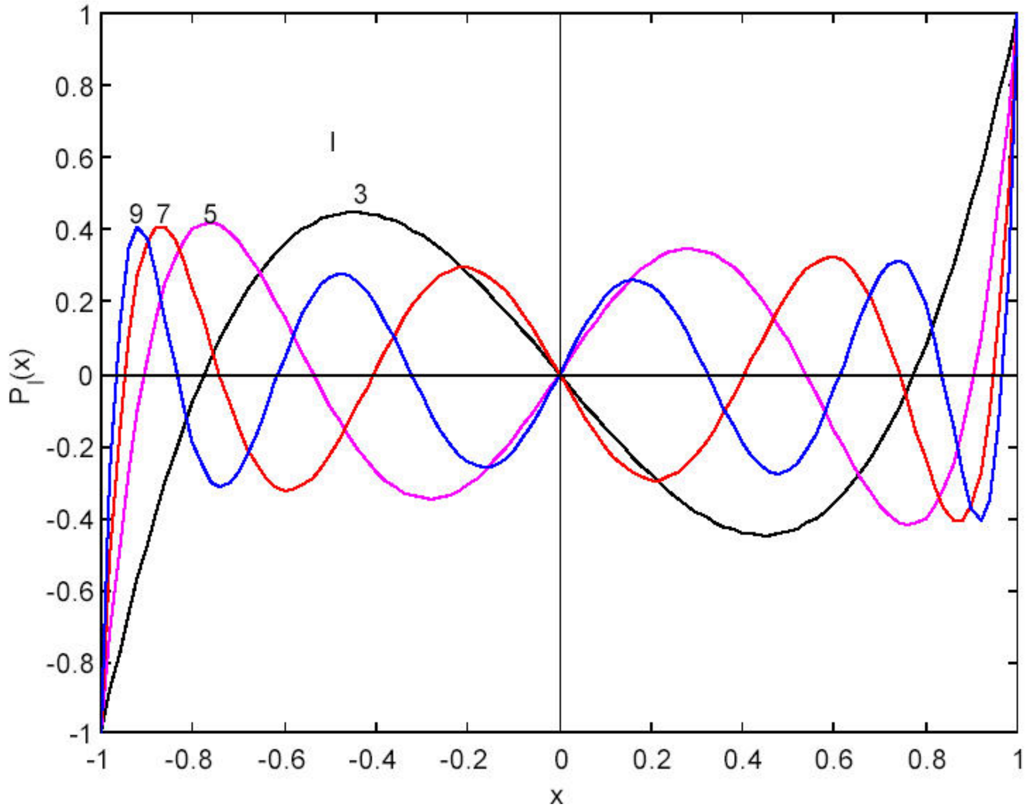


Figure 2 Legendre polynomials for odd n

For further interest, it should be easy to verify, by substitution, that the Legendre polynomials are solutions of the differential equation

$$(1 - x^2)\, y'' - 2xy' + n\,(n + l)\, y = 0 \rightarrow 1.1.6$$

**1.2 Fitting a Polynomial to a Set of Points. Lagrange polynomials. Lagrange Interpolation**

Given a set of n points on a graph, there any many possible polynomials of sufficiently high degree that go through all n of the points. There is, however, just one polynomial of degree less than *n* that will go through them all. Most readers will find no difficulty in determining the polynomial. For example, consider the three points (1, 1), (2, 2), (3, 2). To find the polynomial $y = a_o + a_1x + a_2x^2$ that goes through them, we simply substitute the three points in turn and hence set up the three simultaneous equations

$$\begin{aligned} 1 &= a_0 + a_1 + a_2 \\ 2 &= a_0 + 2a_1 + 4a_2 \rightarrow 1.2.1 \\ 3 &= a_0 + 3a_1 + 9a_2 \end{aligned}$$

and solve them for the coefficients. Thus $a_0 = -1$, $a_1 = 2.5$ and $a_3 = -0.5$. In a similar manner we can fit a polynomial of degree n-1 to go exactly through *n* points. If there are *more than n* points, we may wish to fit a *least squares polynomial* of degree *n* - 1 to go close to the points. We are interested in fitting a polynomial of degree n-1 exactly through *n* points, and we are going to show how to do this by means of Lagrange polynomials.

While the smallest-degree polynomial that goes through *n* points is usually of degree *n* - 1, it could be less than this. For example, we might have four points, all of which fit exactly on a parabola (degree two). However, in general one would expect the polynomial to be of degree n-1, and, if this is not the case, all that will happen is that we shall find that the coefficients of the highest powers of *x* are zero.

That was straightforward. However, what we are going to do in this section is to fit a polynomial to a set of points by using some functions called *Lagrange polynomials*. These are functions that are described by Max Fairbairn as “cunningly engineered” to aid with this task.

Let us suppose that we have a set of *n* points:

$(x_1, y_1), (x_1, y_1), (x_2, y_2), \ldots \ldots (x_i, y_i), \ldots \ldots (x_n, y_n)$,

and we wish to fit a polynomial of degree n-1 to them.

we assert that the function

$$y = \sum_{i=1}^{n} y_i L_i(x) \rightarrow 1.2.2$$

is the required polynomial, where the n functions $L_i(x)$, i = 1, n, are n *Lagrange polynomials*, which are polynomials of degree n-1 defined by

$$L_i(x) = \prod_{\substack{j=`1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \rightarrow 1.2.3$$

Written more explicitly, the first three Lagrange polynomials are

$$L_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)......(x-x_n)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)......(x_1-x_n)} \rightarrow 1.2.4$$

$$L_2(x) = \frac{(x-x_2)(x-x_3)(x-x_4)......(x-x_n)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)......(x_2-x_n)} \rightarrow 1.2.5$$

$$L_3(x) = \frac{(x-x_2)(x-x_3)(x-x_4)......(x-x_n)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)......(x_3-x_n)} \rightarrow 1.2.6$$

At first encounter, this will appear meaningless, but with a simple numerical example it will become clear what it means and also that it has indeed been cunningly engineered for the task.

Consider the same points as before, namely (1, 1), (2, 2), (3, 2). The three Lagrange polynomials are

$$L_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6), \rightarrow 1.2.7$$

$$L_2(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -x^2 + 4x - 3, \rightarrow 1.2.8$$

$$L_3(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2), \rightarrow 1.2.9$$

Equation 1.2.2 for the polynomial of degree n-1 that goes through the three points is, then,

$$y = 1*0.5(x^2-5x+6) + 2*(-x^2+4x-3) + 2*0.5(x^2-3x+2); \rightarrow 1.2.10$$

that is

$$y = -0.5x^2 + 2.5x - 1 \rightarrow 1.2.11$$

which agrees with what we got before.

One way or another, if we have found the polynomial that goes through the *n* points, we can then use the polynomial to interpolate between non tabulated points. Thus we can either determine the coefficients in $y = a_0 + a_1x + a_2x^2 \cdots$ by solving n simultaneous equations, or we can use equation 1.2.2 directly for our interpolation (without the need to calculate the coefficients $a_0$ , $a_1$, etc.), in which case the technique is known as *Lagrangian interpolation.* If the tabulated function for which we need an interpolated value is a polynomial of degree less than *n*, the interpolated value will be exact. Otherwise it will be approximate.

**1.3 The Algorithm of Gaussian Quadrature**

Gaussian quadrature is an alternative method of numerical integration which is often much faster and more spectacular than Simpson's rule. Gaussian quadrature allows you to carry out the integration

$$\int_{-1}^{1} f(x)dx \rightarrow 1.3.1$$

But what happens if our limits of integration are not ±1? What if we want to integrate

$$\int_{a}^{b} F(t)dt? \rightarrow 1.3.2$$

That is no problem at all – we just make a change of variable. Thus, let

$$x = \frac{2t-a-b}{b-a}, t = \frac{1}{2}[(b-a)x+a+b] \rightarrow 1.3.3$$

and the new limits are then $x = \pm 1$.

I now assert, without derivation (see the derivation in section 1.4), that

$$I = \sum_{i=1}^{5} c_{5,i} f(x_{5,i}), \rightarrow 1.3.4$$

Let's try it.

| $x_{5,i}$ | $f(x_{5,i})$ | $c_{5,i}$ |
|---|---|---|
| +0.906 179 845 939 | 0.783 266 908 39 | 0.236 926 885 06 |
| +0.538 469 310 106 | 0.734 361 739 69 | 0.478 628 670 50 |
| 0.000 000 000 000 | 0.555 360 367 27 | 0.568 888888 89 |
| −0.538 469 310 006 | 0.278 501 544 60 | 0.478 628 670 50 |
| −0.906 179 845 939 | 0.057 820 630 35 | 0.236 926 885 06 |

and the expression 1.3.4 comes to 1.000 000 000 04, and might presumably have come even closer to 1 had we given $x_{n,i}$ , and $C_{n,i}$ to more significant figures.

**1.4 Gaussian Quadrature - Derivation**

In order to understand why Gaussian quadrature works so well, we first need to understand some properties of polynomials in general and of Legendre polynomials in particular. We also need to remind ourselves of the use of Lagrange polynomials for approximating an arbitrary function.

First, a statement concerning polynomials in general: Let P be a polynomial of degree *n*, and let *S* be a polynomial of degree less than 2n. Then, if we divide S by P, we obtain a quotient Q and a remainder R, each of which is a polynomial of degree less than n.

That is to say:

$$\frac{S}{P} = Q + \frac{R}{P} \rightarrow 1.4.1$$

What this means is best understood by looking at an example, with n = 3, for example,

Let

$$P=5x^3 - 2x^2 + 3x+7, \rightarrow 1.4.2$$

And

$$S=9x^5+4x^4-5x^3+6x^2+2x-3 \rightarrow 1.4.3$$

If we carry out the division S/P the ordinary process of long division we obtain

$$\frac{9x^5+4x^4-5x^3+6x^2+2x-3}{5x^3-2x^2+3x+7} = 1.8x^2+1.52x-1.472-\frac{14.104x^2+4.224x-7.304}{5x^3-2x^2+3x+7} \rightarrow 1.4.4$$

For example if x = 3, this becomes

$$\frac{2433}{133} = 19.288 - \frac{132.304}{133}$$

The theorem given by equation 1.1.1 is true for any polynomial P of degree n. In particular, it is true if P is the Legendre polynomial of degree n.

Next an important property of the Legendre polynomials namely, if $P_n$ and $P_m$ are Legendre polynomials of degree n, and m respectively then

$$\int_{-1}^{1} P_n P_m dx = 0 \text{ unless m=n.} \rightarrow 1.4.5$$

This property is called the orthogonal property of the Legendre polynomials.

Although the proof is straightforward, it may look formidable at first.

From the symmetry of the Legendre polynomials (see figure 1.1), the following are obvious:

$$\int P_n P_m dx \neq 0 \text{ if m=n}$$

and

$$\int_{-1}^{1} P_n P_m dx = 0 \text{ if one (but not both) of m or n is odd.}$$

In fact we can go further, and as we shall show,

$$\int_{-1}^{1} P_n P_m dx = 0$$ unless m=n, weather m and n are even or odd.

Thus $P_m$ satisfied the differential equation (see equation 1.1.6)

$$(1-x^2)\frac{d^2P_m}{dx_2} - 2x\frac{dP_m}{dx} + m(m+1)P_m = 0 \rightarrow 1.4.6$$

Which can also be written

$$\frac{d}{dx}[(1-x^2)\frac{dP_m}{dx}] + m(m+1)P = 0 \rightarrow 1.4.7$$

Mulitpy by $P_n$

$$P_n\frac{d}{dx}[(1-x^2)\frac{dP_m}{dx}] + m(m+1)P_m P_n = 0 \rightarrow 1.4.8$$

Which can also be written

$$\frac{d}{dx}[(1-x^2)P_n\frac{dP_m}{dx}] - (1-x^2)\frac{dP_n}{dx}\frac{dP_m}{dx} + m(m+1)P_m P_n + 0 \rightarrow 1.4.9$$

In a similar manner, we have

$$\frac{d}{dx}[(1-x^2)(P_n\frac{dP_m}{dx} - P_m\frac{dP_n}{dx})] + [m(m+1) - n(n+1)]P_m P_n = 0 \rightarrow 1.4.10$$

Subtract one form the other:

$$\frac{d}{dx}[(1-x^2)(P_n\frac{dP_n}{dx} - P_m\frac{dP_m}{dx})] + [m(m+1) - n(n+1)]P_m P_n = 0 \rightarrow 1.4.11$$

Integrate from -1 to +1:

$$[(1-x^2)(P_n\frac{dP_m}{dx} - P_m\frac{dP_n}{dx})] = [n(n+1) - m(m+1)]\int_{-1}^{1} P_m P_n dx \rightarrow 1.4.12$$

The left hand side is zero because $1-x^2$ is zero at both limits.
There for, unless m=n

$$\int_{-1}^{1} P_n P_m dx = 0 \rightarrow 1.4.13$$

We now assert that, if $P_n$ is the Legendre polynomial of degree n, and if Q is any polynomial of degree less than n, then

$$\int_{-1}^{1} P_n Q dx = 0 \rightarrow 1.4.14$$

We shall first prove this, and then give and example to see what it means.

To start the proof we recall the recursion relation (see equation 1.1.4-through here we are substituting n-1 for n) for the Legendre polynomials:

$$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2} \rightarrow 1.4.15$$

The proof will be induction.

Let Q be any polynomial of degree les than n. Multiply the above relation by Qdx and integrate from -1 to +1:

$$n\int_{-1}^{1} P_n Qdx = (2n-1)\int_{-1}^{1} xP_{n-1}Qdx - (n-1)\int P \rightarrow 1.4.16$$

If the right hand side is zero, then the left hand side is also zero.
For example, let n=4, so that

$$P_{n-2}=P_2=0.5(3x^2-1) \rightarrow 1.4.17$$

and

$$xP_{n-1}=xP_3=0.5(5x^4-3x^2), \rightarrow 1.4.18$$

and let

$$Q=2(a_3x^3+a_2x^2+a_1x+a_0) \rightarrow 1.4.19$$

It is then straightforward (and only slightly tedious) to show that

$$\int_{-1}^{1} P_{n-2}Qdx = (\frac{6}{5}-\frac{2}{3})a_2 \rightarrow 1.4.20$$

And that

$$\int_{-1}^{1} xP_{n-1}Qdx = (\frac{10}{7}-\frac{6}{5})a_2 \rightarrow 1.4.21$$

But

$$7(\frac{10}{7}-\frac{6}{5})a_2 - 3(\frac{6}{5}-\frac{2}{3})a_2 = 0 \rightarrow 1.4.22$$

And therefore

$$\int_{-1}^{1} P_4 Qdx = 0 \rightarrow 1.4.23$$

We have shown that

$$n\int_{-1}^{1} P_n Qdx = (2n-1)\int_{-1}^{1} xP_{n-1}Qdx - (n-1)\int_{-1}^{1} P_{n-2}Qdx = 0 \rightarrow 1.4.24$$

For n=4, and therefore it is true for all positive integral n.

We can use this property for a parlour trick. For example, you can say "Think of any polynomial. Don't tell me what it is –Just tell me its degree. Then multiply it by (here give a Legendre polynomial of degree more than this). Now integrate form -1 to +1. The answer is zero right?" (Applause).

Thus: Think of any polynomial. $3x^2$-5x+7 . Now multiply it by $5x^3$ - 3x. Ok, that's $15x^5 - 25x^4 - 2x^3 + 15x^2 - 21x$. Now integrate it from -1 to +1. The answer is zero.

Now, let S be any polynomial of degree less than 2n. Let us divide it b the Legendre polynomial of degree n, $P_n$ to obtain the quotient Q and a remainder R, both of degree less than n. Then n assert that

$$\int_{-1}^{1} Sdx = \int_{-1}^{1} Rdx \rightarrow 1.4.25$$

This follows trivially from equation 1.4.1 and 1.4.14. Thus

$$\int_{-1}^{1} Sdx = \int_{-1}^{1} (QP_1 + R)dx = \int_{-1}^{1} Rdx \rightarrow 1.4.26$$

Example:

Let $S=6x^5-12x^4+4x^3+7x^2-5x+7$.

The integral of this from -1 to +1 is 13.86. If we divide S by $0.5(5x^3-3x)$, we obtain a quotient of $2.4x^2-4.8x+3.04$ and a remainder of $-0.2x^2-0.44x+7$. The integral of the latter from -1 to +1 is also 13.86

I have just described some properties of Legendre polynomials. Before getting on to the rationale behind Gaussian quadrature, let us remind ourselves from Section 1.2 about Lagrange polynomials. We recall from that section that, if we have a set of *n* points, the following function:

$$y = \sum_{i=1}^{n} y_i L_i(x) \rightarrow 1.4.27$$

(in which the *n* functions $L_i(x)$, i=, n, are Lagrange polynomials of degree n-1) is the polynomial of degree n-1 that passes exactly through the *n* points. Also, if we have some function *f*(*x*) which we evaluate at *n* points, then the polynomial

$$y = \sum_{i=1}^{n} f(x_i) L_i(x) \rightarrow 1.4.28$$

is a jolly good approximation to *f*(*x*) and indeed may be used to interpolated between nontabulated points, even if the function is tabulated at irregular intervals. In particular, if *f*(*x*) is a polynomial of degree n-1, then the expression 1.3.28 is an exact representation of *f*(*x*).

We are now ready to start talking about quadrature. We finish to approximate $\int_{-1}^{1} f(x)dx$ by and n-term finite series

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} C_i f(x_i), \rightarrow 1.4.29$$

Where $-1 < x_i < 1$. To this end, we can approximate *f*(*x*) by the right hand side of equation s1.4.28, so that

$$\int_{-1}^{1} f(x)dx \approx \int_{-1}^{1}\sum_{i=1}^{n} f(x_i)L_i(x)dx = f(x_i)\int_{-1}^{1}\sum_{i=1}^{n} L_i(x)dx. \rightarrow 1.4.30$$

Recall that the Lagrange polynomials in this expression are of degree n-1.
The required coefficients for equation 1.4.29 are therefore

$$C_i = \int_{-1}^{1} L_i(x)dx. \rightarrow 1.4.31$$

Note that at this stage the values of the $x_i$ have not yet been chosen; they are merely restricted to the interval [-1 , 1].

Now let's consider $\int_{-1}^{1} S(x)dx,$ where *S* is a polynomial of degree less than 2*n*, such as, for example, the polynomial of equation 1.4.3. We can write

$$\int_{-1}^{1} S(x)dx = \int_{-1}^{1}\sum_{i=1}^{n} S(x_i)L_i(x)dx = \int_{-1}^{1}\sum_{i=1}^{n} L_i(x)[Q(x_i)P(x_i) + R(x_i)]dx \quad \rightarrow 1.4.32$$

Here, as before, *P* is a polynomial of degree *n*, and *Q* and *R* are of degree less than *n*. If we now choose the *xi* to be the roots of the Legendre polynomials, then

$$\int_{-1}^{1} S(x)dx = \int_{-1}^{1}\sum_{i=1}^{n} L_i(x)R(x_i)dx \rightarrow 1.3.33$$

Note that the integrand on the right hand side of equation 1.16.33 is an *exact representation of R*(*x*). But we have already shown (equation 1.4.26) that

$$\int_{-1}^{1} S(x)dx = \int_{-1}^{1} R(x)dx = \sum_{i=1}^{n} C_i R(x_i) = \sum_{i=1}^{n} C_i S(x_i) \quad \rightarrow 1.4.34$$

It follows that the Gaussian quadrature method, if we choose the roots of the Legendre polynomials for the *n* abscissas, will yield exact results for any polynomial of degree less than 2*n*, and will yield a good approximation to the integral if *S*(*x*) is a polynomial representation of a general function *f*(*x*) obtained by fitting a polynomial to several points on the function.

Roots of $P_l = 0$

| $l$ | $x_{l,i}$ | $c_{l,i}$ |
|---|---|---|
| 2 | $\pm$0.577 350 269 190 | 1.000 000 000 00 |
| 3 | $\pm$0.774 596 669 241 | 0.555 555 555 56 |
| | 0.000 000 000 000 | 0.888 888 888 89 |
| 4 | $\pm$0.861 136 311 594 | 0.347 854 845 14 |
| | $\pm$0.339 981 043 585 | 0.652 145 154 86 |
| 5 | $\pm$0.906 179 845 939 | 0.236 926 885 06 |
| | $\pm$0.538 469 310 106 | 0.478 628 670 50 |
| | 0.000 000 000 000 | 0.568 888 888 89 |
| 6 | $\pm$0.932 469 514 203 | 0.171 324 492 38 |
| | $\pm$0.661 209 386 466 | 0.360 761 573 05 |
| | $\pm$0.238 619 186 083 | 0.467 913 934 57 |
| 7 | $\pm$0.949 107 912 343 | 0.129 484 966 17 |
| | $\pm$0.741 531 185 599 | 0.279 705 391 49 |
| | $\pm$0.405 845 151 377 | 0.381 830 050 50 |
| | 0.000 000 000 000 | 0.417 959 183 68 |

| $l$ | $x_{l,i}$ | $c_{l,i}$ |
|---|---|---|
| 8 | ±0.960 289 856 498 | 0.101 228 536 29 |
| | ±0.796 666 477 414 | 0.222 381 034 45 |
| | ±0.525 532 409 916 | 0.313 706 645 88 |
| | ±0.183 434 642 496 | 0.362 683 783 38 |
| 9 | ±0.968 160 239 508 | 0.081 274 388 36 |
| | ±0.836 031 107 327 | 0.180 648 160 69 |
| | ±0.613 371 432 701 | 0.260 610 696 40 |
| | ±0.324 253 423 404 | 0.312 347 077 04 |
| | 0.000 000 000 000 | 0.330 239 355 00 |
| 10 | ±0.973 906 528 517 | 0.066 671 343 99 |
| | ±0.865 063 366 689 | 0.149 451 349 64 |
| | ±0.679 409 568 299 | 0.219 086 362 24 |
| | ±0.433 395 394 129 | 0.269 266 719 47 |
| | ±0.148 874 338 982 | 0.295 524 224 66 |
| 11 | ±0.978 228 658 146 | 0.055 668 567 12 |
| | ±0.887 062 599 768 | 0.125 580 369 46 |
| | ±0.730 152 005 574 | 0.186 290 210 93 |
| | ±0.519 096 129 207 | 0.233 193 764 59 |
| | ±0.269 543 155 952 | 0.262 804 544 51 |
| | 0.000 000 000 000 | 0.272 925 086 78 |
| 12 | ±0.981 560 634 247 | 0.047 175 336 39 |
| | ±0.904 117 256 370 | 0.106 939 325 99 |
| | ±0.769 902 674 194 | 0.160 078 328 54 |
| | ±0.587 317 954 287 | 0.203 167 426 72 |
| | ±0.367 831 498 998 | 0.233 492 536 54 |
| | ±0.125 233 408 511 | 0.249 147 045 81 |
| 13 | ±0.984 183 054 719 | 0.040 484 004 77 |
| | ±0.917 598 399 223 | 0.092 121 499 84 |
| | ±0.801 578 090 733 | 0.138 873 510 22 |
| | ±0.642 349 339 440 | 0.178 145 980 76 |
| | ±0.448 492 751 036 | 0.207 816 047 54 |
| | ±0.230 458 315 955 | 0.226 283 180 26 |
| | 0.000 000 000 000 | 0.232 551 553 23 |

| | | |
|---|---|---|
| 14 | ±0.986 283 808 697 | 0.035 119 460 33 |
| | ±0.928 434 883 664 | 0.080 158 087 16 |
| | ±0.827 201 315 070 | 0.121 518 570 69 |
| | ±0.687 292 904 812 | 0.157 203 167 16 |
| | ±0.515 248 636 358 | 0.185 538 397 48 |
| | ±0.319 112 368 928 | 0.205 198 463 72 |
| | ±0.108 054 948 707 | 0.215 263 853 46 |
| 15 | ±0.987 992 518 020 | 0.030 753 242 00 |
| | ±0.937 273 392 401 | 0.070 366 047 49 |
| | ±0.848 206 583 410 | 0.107 159 220 47 |
| | ±0.724 417 731 360 | 0.139 570 677 93 |
| | ±0.570 972 172 609 | 0.166 269 205 82 |
| | ±0.394 151 347 078 | 0.186 161 000 02 |
| | ±0.201 194 093 997 | 0.198 431 485 33 |
| | 0.000 000 000 000 | 0.202 578 241 92 |
| 16 | ±0.989 400 934 992 | 0.027 152 459 41 |
| | ±0.944 575 023 073 | 0.062 253 523 94 |
| | ±0.865 631 202 388 | 0.095 158 511 68 |
| | ±0.755 404 408 355 | 0.124 628 971 26 |
| | ±0.617 876 244 403 | 0.149 595 988 82 |
| | ±0.458 016 777 657 | 0.169 156 519 39 |
| | ±0.281 603 550 779 | 0.182 603 415 04 |
| | ±0.095 012 509 838 | 0.189 450 610 46 |
| 17 | ±0.990 575 475 315 | 0.024 148 302 87 |
| | ±0.950 675 521 769 | 0.055 459 529 38 |
| | ±0 880 239 153 727 | 0.085 036 148 32 |
| | ±0.781 514 003 897 | 0.111 883 847 19 |
| | ±0.657 671 159 217 | 0.135 136 368 47 |
| | ±0.512 690 537 086 | 0.154 045 761 08 |
| | ±0.351 231 763 454 | 0.168 004 102 16 |
| | ±0.178 484 181 496 | 0.176 562 705 37 |
| | 0.000 000 000 000 | 0.179 446 470 35 |

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