ChE 548
Final Exam
Spring, 2004

Problem 1.
Consider a single-component, incompressible fluid moving down an uninsulated funnel. Derive the energy balance for this system. Show all work involved in each step of the derivation. Express the energy balance in such a form that the left-hand-side contains only the time derivative of the temperature. State any assumptions that you make. Introduce variables such as the density, heat capacity, thermal conductivity, etc as necessary. The fact that the fluid is incompressible can be expressed by making the velocity a function of axial position; do so. Assume the surroundings are hotter than the fluid inside the funnel. Qualitatively sketch the steady state profile for two values of the heat transfer coefficient, zero (insulated) and non-zero for your boundary conditions. For the insulated case, one can obtain an analytical solution for the steady state profile. Time permitting, obtain it.

Problem 1. Solution:

mass balance:

$$\frac{\partial \rho}{\partial t} = 0 \qquad \text{for incompressible fluid.}$$

density is constant.

enthalpy

assume constant heat capacity

assume heat capacity is given on a per mass basis

$$H = \int_{T_{ref}}^{T} C_p dT = C_p \left(T - T_{ref}\right)$$

length of funnel = L

diameter of the reactor

$$D_R(z) = D_R(0) + \frac{D_R(L) - D_R(0)}{L} z$$

cross-sectional area

$$A_x(z) = \frac{\pi}{4} D_R(z)^2 = \frac{\pi}{4}\left(D_R(0) + \frac{D_R(L) - D_R(0)}{L} z\right)^2$$

for an incompressible fluid, the volumetric flowrate is constant

$$F = \text{constant} = v(z) A_x(z)$$

Therefore, the velocity as a function of position is

$$v(z) = \frac{F}{A_x(z)}$$

volume element

assume no variation in radial or angular dimensions

$$\Delta V = A_x \Delta z$$

energy balance

$$acc = \Delta V \frac{\partial \rho H}{\partial t}$$

$$conv = \rho v H A_x \big|_z - \rho v H A_x \big|_{z+\Delta z}$$

$$cond = q A_x \big|_z - q A_x \big|_{z+\Delta z}$$

$$loss = -A_s h(T - T_{surr}) = -\Delta z \pi D_R(z) h(T - T_{surr})$$

where T_{surr} is the temperature of the surroundings and h is the heat transfer coefficient.

$$\Delta V \rho H = \rho v H A_x \big|_z - \rho v H A_x \big|_{z+\Delta z} + q A_x \big|_z - q A_x \big|_{z+\Delta z} - \Delta z \pi D_R(z) h(T - T_{surr})$$

divide by incremental volume

$$\frac{\partial \rho H}{\partial t} = \frac{1}{A_x}\left[\frac{\rho v H A_x \big|_z - \rho v H A_x \big|_{z+\Delta z}}{\Delta z}\right] + \frac{1}{A_x}\left[\frac{q A_x \big|_z - q A_x \big|_{z+\Delta z}}{\Delta z}\right] - \frac{4}{D_R(z)} h(T - T_{surr})$$

take limit as Δz goes to zero.

$$\frac{\partial \rho H}{\partial t} = -\frac{1}{A_x}\left[\frac{\partial \rho v H A_x}{\partial z}\right] - \frac{1}{A_x}\left[\frac{\partial q A_x}{\partial z}\right] - \frac{4}{D_R(z)} h(T - T_{surr})$$

insert Fourier's Law

$$\frac{\partial \rho H}{\partial t} = -\frac{1}{A_x}\left[\frac{\partial \rho v H A_x}{\partial z}\right] + \frac{1}{A_x}\left[\frac{\partial}{\partial z}\left(k_c A_x \frac{\partial T}{\partial z}\right)\right] - \frac{4}{D_R(z)} h(T - T_{surr})$$

eliminate the enthalpy in favor of the temperature

$$\frac{\partial \rho H}{\partial t} = \rho C_p \frac{\partial T}{\partial t}$$

$$\frac{\partial \rho v H A_x}{\partial z} = \rho\left[v A_x \frac{\partial H}{\partial z} + H \frac{\partial v A_x}{\partial z}\right] = \rho C_p \left[v A_x \frac{\partial T}{\partial z} + (T - T_{ref}) \frac{\partial v A_x}{\partial z}\right]$$

assume constant thermal conductivity.
use the fact that we have an expression for the velocity as a function fo axial position.

$$\frac{\partial \rho v H A_x}{\partial z} = \rho C_p F \frac{\partial T}{\partial z}$$

$$\rho C_p \frac{\partial T}{\partial t} = -\rho C_p v \frac{\partial T}{\partial z} + k_c \frac{\partial^2 T}{\partial z^2} + \frac{k_c}{A_x}\frac{\partial T}{\partial z}\frac{\partial A_x}{\partial z} - \frac{4}{D_R(z)} h(T - T_{surr})$$

divide by ρC_p

$$\frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial z} + \alpha \frac{\partial^2 T}{\partial z^2} + \alpha \frac{\partial T}{\partial z}\frac{\partial \ln A_x}{\partial z} - \frac{4}{\rho C_p D_R(z)} h(T - T_{surr})$$

where α is the thermal diffusivity

$$\alpha = \frac{k_c}{\rho C_p}$$

$$\frac{\partial \ln A_x}{\partial z} = \frac{1}{A_x}\frac{\partial A_x}{\partial z} = \frac{1}{A_x}\frac{\partial}{\partial z}\frac{\pi}{4}\left(D_R(0) + \frac{D_R(L) - D_R(0)}{L} z\right)^2$$
$$= \frac{1}{A_x}\frac{\pi}{2}\left(D_R(0) + \frac{D_R(L) - D_R(0)}{L} z\right)\frac{D_R(L) - D_R(0)}{L}$$
$$= 2\frac{D_R(L) - D_R(0)}{L\left(D_R(0) + \frac{D_R(L) - D_R(0)}{L} z\right)} = 2\frac{D_R(L) - D_R(0)}{L D_R(z)}$$

$$\frac{\partial T}{\partial t} = -v \frac{\partial T}{\partial z} + \alpha \frac{\partial^2 T}{\partial z^2} + \alpha \frac{\partial T}{\partial z} 2\frac{D_R(L) - D_R(0)}{L D_R(z)} - \frac{4}{\rho C_p D_R(z)} h(T - T_{surr})$$

This is the evolution equation for temperature.

Now, let's solve for the steady state profile for the insulated case, where h=0.

Case A. The temperature and the temperature gradient at z=0 are known.

$$0 = -\frac{F}{A_x(z)}\frac{\partial T}{\partial z} + \alpha\frac{\partial^2 T}{\partial z^2} + \alpha\frac{\partial T}{\partial z}2\frac{D_R(L)-D_R(0)}{LD_R(z)} \quad (A.1)$$

$$0 = \left(2\alpha\frac{D_R(L)-D_R(0)}{LD_R(z)} - \frac{F}{A_x(z)}\right)\frac{\partial T}{\partial z} + \alpha\frac{\partial^2 T}{\partial z^2} \quad (A.2)$$

$$0 = \left(2\alpha\frac{D_R(L)-D_R(0)}{LD_R(z)} - \frac{4F}{\pi D_R(z)^2}\right)\frac{\partial T}{\partial z} + \alpha\frac{\partial^2 T}{\partial z^2} \quad (A.3)$$

let $u = \frac{\partial T}{\partial z}$ (A.4)

$$0 = \left(2\alpha\frac{D_R(L)-D_R(0)}{LD_R(z)} - \frac{4F}{\pi D_R(z)^2}\right)u + \alpha\frac{\partial u}{\partial z} \quad (A.5)$$

This ODE is of the form:

$$a(z)u + \frac{\partial u}{\partial z} = 0 \quad (A.6)$$

where $a(z) = 2\frac{D_R(L)-D_R(0)}{LD_R(z)} - \frac{4F}{\alpha\pi D_R(z)^2}$ (A.7)

This ODE has the solution:

$$u(z) = u(z_o)\exp\left(-\int_{z_o}^{z} a(z)dz\right) \quad (A.8)$$

$$D_R(z) = D_R(0) + \frac{D_R(L)-D_R(0)}{L}z \quad (A.9)$$

$$\frac{L}{D_R(L)-D_R(0)}dD_R(z) = dz \quad (A.10)$$

$$\begin{aligned}\int_{z_o}^{z} a(z)dz &= \int_{z_o}^{z}\left[2\frac{D_R(L)-D_R(0)}{LD_R(z)}-\frac{4F}{\alpha\pi D_R(z)^2}\right]dz \\ &= \int_{D_R(0)}^{D_R(z)}\left[2\frac{D_R(L)-D_R(0)}{LD_R(z)}-\frac{4F}{\alpha\pi D_R(z)^2}\right]\frac{L}{D_R(L)-D_R(0)}dD_R(z) \\ &= \int_{D_R(0)}^{D_R(z)}\left[\frac{2}{D_R(z)}-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi D_R(z)^2}\right]dD_R(z) \\ &= 2\ln\left(\frac{D_R(z)}{D_R(0)}\right)+\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\end{aligned} \tag{A.11}$$

$$\begin{aligned}u(z) &= u(z_o)\exp\left(-2\ln\left(\frac{D_R(z)}{D_R(0)}\right)-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right) \\ &= u(z_o)\left(\frac{D_R(0)}{D_R(z)}\right)^2\exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right)\end{aligned} \tag{A.12}$$

$$\frac{\partial T}{\partial z} = \left.\frac{\partial T}{\partial z}\right|_{z=0}\left(\frac{D_R(0)}{D_R(z)}\right)^2\exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right) \tag{A.13}$$

integrate again:

$$\int_{T(0)}^{T(z)} dT = \int_{z_o}^{z}\left.\frac{\partial T}{\partial z}\right|_{z=0}\left(\frac{D_R(0)}{D_R(z)}\right)^2\exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right)dz \tag{A.14}$$

$$T(z) = T(0)+\int_{z_o}^{z}\left.\frac{\partial T}{\partial z}\right|_{z=0}\left(\frac{D_R(0)}{D_R(z)}\right)^2\exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right)dz \tag{A.15}$$

$$\begin{aligned}T(z) &= T(0)+\int_{D_R(0)}^{D_R(z)}\left.\frac{\partial T}{\partial z}\right|_{z=0}\left(\frac{D_R(0)}{D_R(z)}\right)^2\exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right)\frac{L}{D_R(L)-D_R(0)}dD_R(z) \\ &= T(0)+\left.\frac{\partial T}{\partial z}\right|_{z=0}\frac{L}{D_R(L)-D_R(0)}\int_{D_R(0)}^{D_R(z)}\left(\frac{D_R(0)}{D_R(z)}\right)^2\exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right)dD_R(z)\end{aligned} \tag{A.16}$$

This integral is of the form:

$$\int_{x_o}^{x} \frac{1}{x^2} \exp\left(\frac{a}{x} + b\right) dx \tag{A.17}$$

Use the substitution u=1/x and we have

$$\int_{u_o}^{u} u^2 \exp(au + b) dx \tag{A.18}$$

This integral can be evaluated analytically to yield

$$\int_{u_o}^{u} u^2 \exp(au + b) dx = \exp(b)\left[\exp(au)\left(u^2 - 2\frac{u}{a} + \frac{2}{a^2}\right)\right]_{u_o}^{u} \tag{A.19}$$

So for our case

$$u = \frac{1}{D_R(z)} \tag{A.20}$$

$$b = \frac{L}{D_R(L) - D_R(0)} \frac{4F}{\alpha\pi} \frac{1}{D_R(0)} \tag{A.21}$$

$$a = -\frac{L}{D_R(L) - D_R(0)} \frac{4F}{\alpha\pi} \tag{A.22}$$

So we have

$$\int_{u_o}^{u} u^2 \exp(au + b) dx = \exp(b)\left[\exp\left(\frac{a}{D_R(z)}\right)\left(\frac{1}{D_R(z)^2} - 2\frac{1}{aD_R(z)} + \frac{2}{a^2}\right)\right]_{u_o}^{u} \tag{A.23}$$

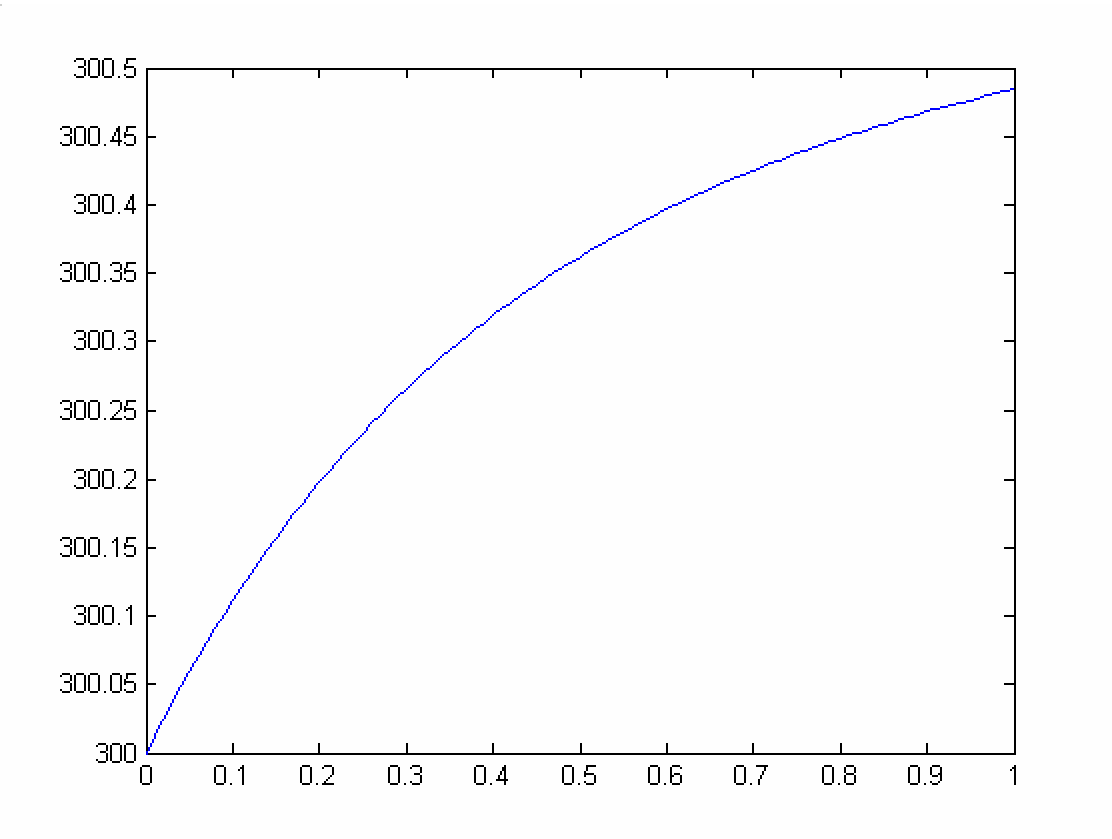
$$T(z) = T(0) + \left.\frac{\partial T}{\partial z}\right|_{z=0} \frac{LD_R(0)^2}{D_R(L) - D_R(0)} \exp(b)\left[\exp\left(\frac{a}{D_R(z)}\right)\left(\frac{1}{D_R(z)^2} - 2\frac{1}{aD_R(z)} + \frac{2}{a^2}\right)\right]_{D_R(0)}^{D_R(z)} \tag{A.24}$$

This solution assumes that we know the temperature and the temperature gradient at the inlet of the funnel. We could also work the problem out where our constants of integration are determined by a temperature at each boundary.

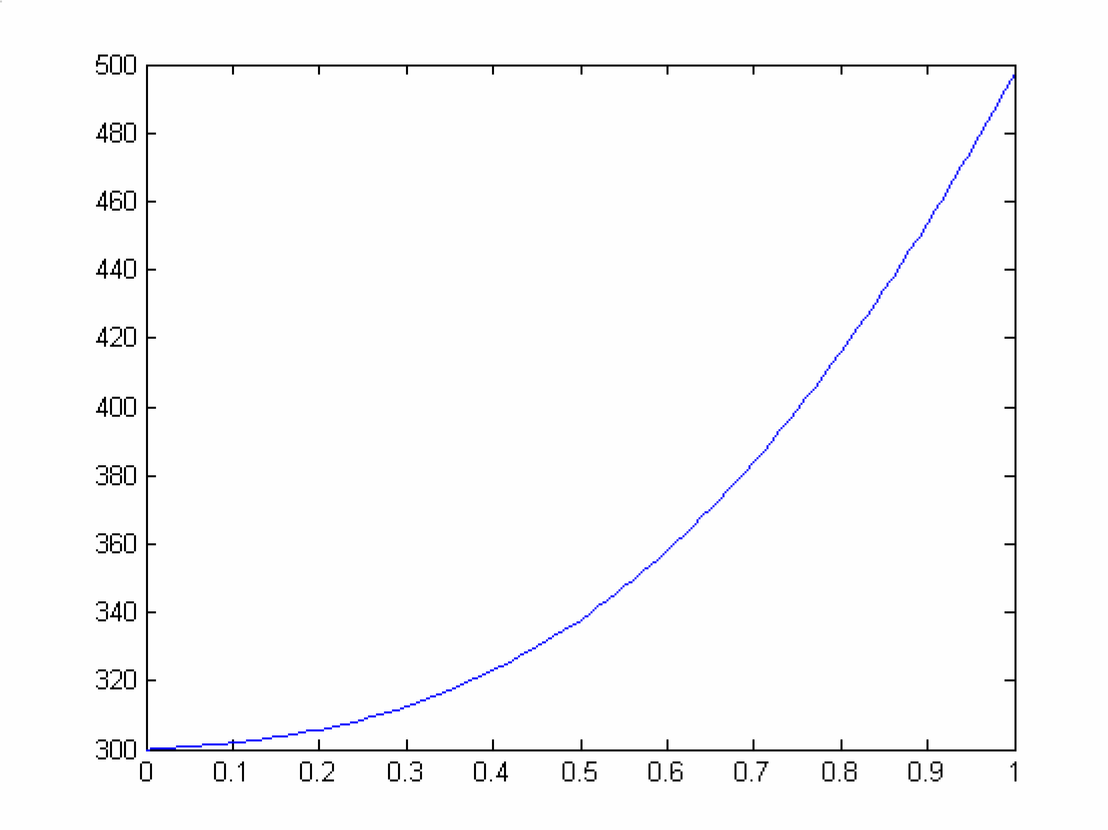
There you have the analytical solution.
Let's make a couple plots of the analytical solution.
First, we write a quick little Matlab code to evaluate the solution.

```
clear all;
close all;
nzint = 100;
nzp = nzint + 1;
T = zeros(1,nzp);
z = zeros(1,nzp);
L = 1;
zo = 0;
zf = zo + L;
dz = L/nzint;
Cp = 1;
Dro = 1;
DrL = 2;
Ds = (DrL - Dro)/L;
F = 1;
rho = 1;
kc = 1;
alpha = kc/(rho*Cp);
dTdzo = 1;
To = 300;
a = -4*F/(alpha*pi*Ds);
b = 4*F/(alpha*pi*Ds*Dro);
expb = exp(b);
term1 = exp(a/Dro)*(1/Dro^2 - 2/(a*Dro) + 2/a^2);
for i = 1:1:nzp
   z(i) = (i-1)*dz + zo;
   Dr = Dro + Ds*z(i);
   Ax = pi/4*Dr*Dr;
   v = F/Ax;
   term2 = exp(a/Dr)*(1/Dr^2 - 2/(a*Dr) + 2/a^2);
   T(i) = To + dTdzo*Dro^2/Ds*expb*(term2 - term1);
end
plot(z,T);
```

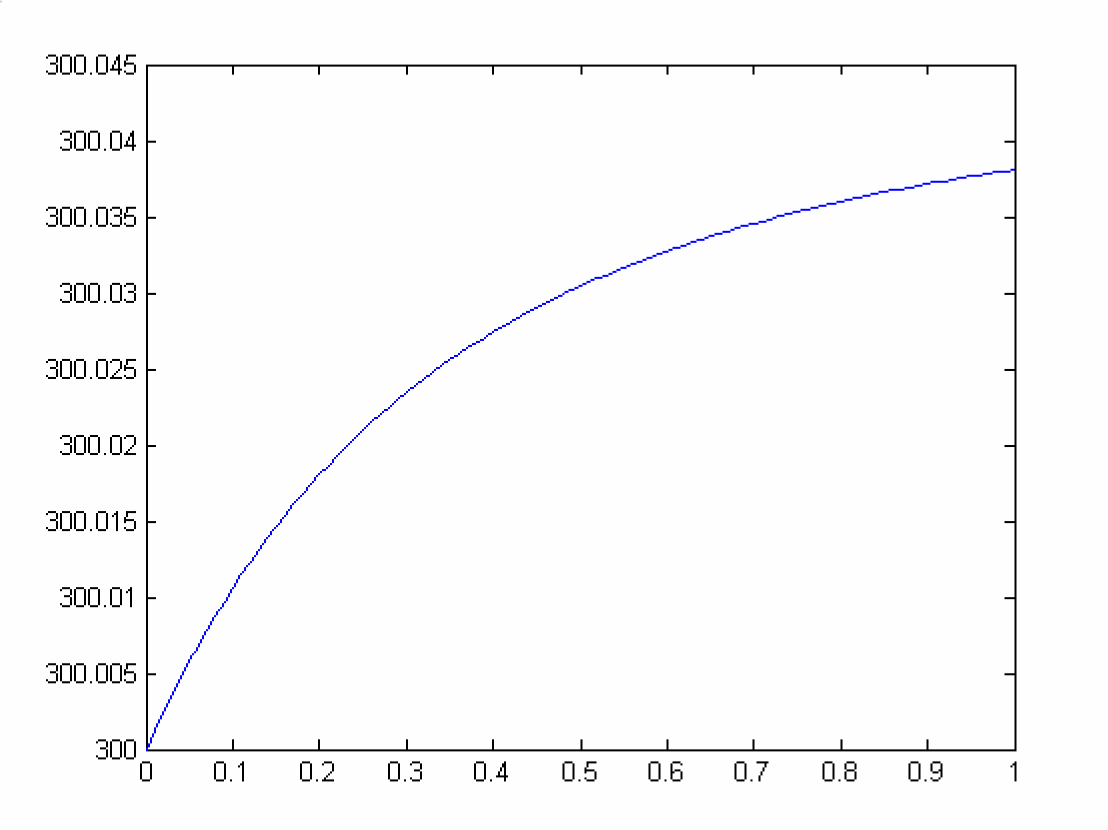
Plot for base case:



plot for base case except we increase the volumetric flowrate to F = 10



plot for base case except we increase the thermal conductivity to kc = 10



Case B. The temperature at z=0 and the temperature at z=L are known.

Equations (A.1) to (A.12) remain the same. Equation (A.12) can be rewritten in terms of a general unknown constant of integration.

$$u(z) = u(z_o)\left(\frac{D_R(0)}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}-\frac{1}{D_R(0)}\right)\right) \quad \text{(A.12)}$$

$$u(z) = c\left(\frac{1}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}\right)\right) \quad \text{(B.12)}$$

$$\frac{\partial T}{\partial z} = c\left(\frac{1}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}\right)\right) \quad \text{(B.13)}$$

integrate again:

$$\int_{T(0)}^{T(z)} dT = \int_{z_o}^{z} c\left(\frac{1}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}\right)\right)dz \quad \text{(B.14)}$$

$$T(z) = T(0) + \int_{z_o}^{z} c\left(\frac{1}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}\right)\right)dz \quad \text{(B.15)}$$

$$\begin{aligned} T(z) &= T(0) + \int_{D_R(0)}^{D_R(z)} c\left(\frac{1}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}\right)\right)\frac{L}{D_R(L)-D_R(0)}dD_R(z) \\ &= T(0) + c\frac{L}{D_R(L)-D_R(0)}\int_{D_R(0)}^{D_R(z)} \left(\frac{1}{D_R(z)}\right)^2 \exp\left(-\frac{L}{D_R(L)-D_R(0)}\frac{4F}{\alpha\pi}\left(\frac{1}{D_R(z)}\right)\right)dD_R(z) \end{aligned} \quad \text{(B.16)}$$

This integral is of the form:

$$\int_{x_o}^{x} \frac{1}{x^2}\exp\left(\frac{a}{x}+b\right)dx \quad \text{(A.17)}$$

Use the substitution u=1/x and we have

$$\int_{u_o}^{u} u^2 \exp(au+b)dx \quad \text{(A.18)}$$

This integral can be evaluated analytically to yield

$$\int_{u_o}^{u} u^2 \exp(au+b)dx = \exp(b)\left[\exp(au)\left(u^2 - 2\frac{u}{a} + \frac{2}{a^2}\right)\right]_{u_o}^{u} \tag{A.19}$$

So for our case

$$u = \frac{1}{D_R(z)} \tag{A.20}$$

$$b = 0 \tag{B.21}$$

$$a = -\frac{L}{D_R(L) - D_R(0)}\frac{4F}{\alpha\pi} \tag{A.22}$$

So we have

$$\int_{u_o}^{u} u^2 \exp(au+b)dx = \left[\exp\left(\frac{a}{D_R(z)}\right)\left(\frac{1}{D_R(z)^2} - 2\frac{1}{aD_R(z)} + \frac{2}{a^2}\right)\right]_{u_o}^{u} \tag{B.23}$$

$$T(z) = T(0) + c\frac{L}{D_R(L) - D_R(0)}\left[\exp\left(\frac{a}{D_R(z)}\right)\left(\frac{1}{D_R(z)^2} - 2\frac{1}{aD_R(z)} + \frac{2}{a^2}\right)\right]_{D_R(0)}^{D_R(z)} \tag{B.24}$$

Finally, we evaluate the unknown constant c, by forcing it to satisfy the second boundary condition

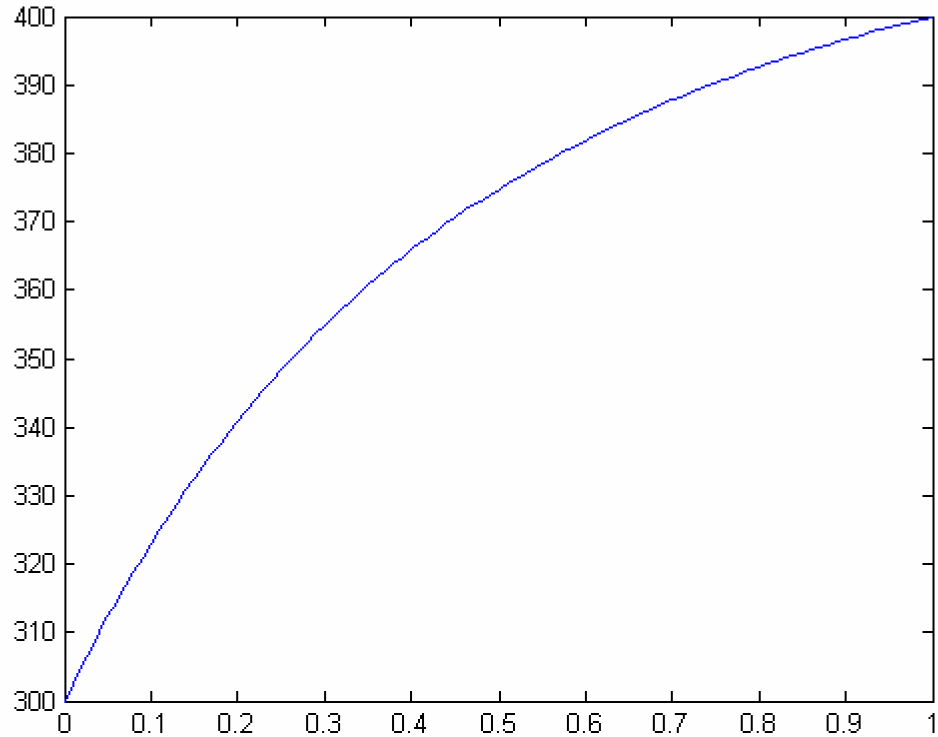
$$T(z=L) = T(0) + c\frac{L}{D_R(L) - D_R(0)}\left[\exp\left(\frac{a}{D_R(z)}\right)\left(\frac{1}{D_R(z)^2} - 2\frac{1}{aD_R(z)} + \frac{2}{a^2}\right)\right]_{D_R(0)}^{D_R(z=L)} \tag{B.25}$$

$$c = \frac{T(z=L) - T(0)}{\frac{L}{D_R(L) - D_R(0)}\left[\exp\left(\frac{a}{D_R(z)}\right)\left(\frac{1}{D_R(z)^2} - 2\frac{1}{aD_R(z)} + \frac{2}{a^2}\right)\right]_{D_R(0)}^{D_R(z=L)}} \tag{B.26}$$

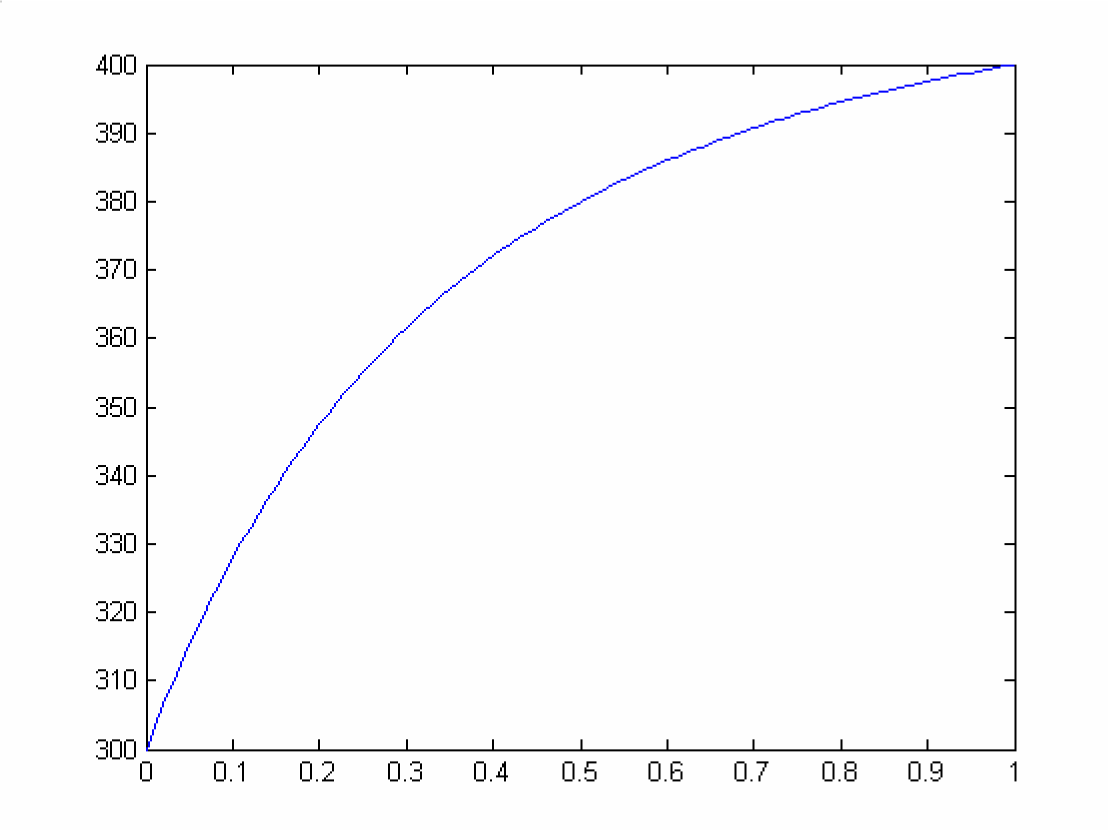
There you have the analytical solution.
Let’s make a couple plots of the analytical solution.
First, we write a quick little Matlab code to evaluate the solution.

```
clear all;
close all;
nzint = 100;
nzp = nzint + 1;
T = zeros(1,nzp);
z = zeros(1,nzp);
L = 1;
zo = 0;
zf = zo + L;
dz = L/nzint;
Cp = 1;
Dro = 1;
DrL = 2;
Ds = (DrL - Dro)/L;
F = 1;
rho = 1;
kc = 1;
alpha = kc/(rho*Cp);
To = 300;
TL = 400;
a = -4*F/(alpha*pi*Ds);
term1 = exp(a/Dro)*(1/Dro^2 - 2/(a*Dro) + 2/a^2);
term3 = exp(a/DrL)*(1/DrL^2 - 2/(a*DrL) + 2/a^2);
c = (TL-To)*Ds/(term3-term1);
for i = 1:1:nzp
   z(i) = (i-1)*dz + zo;
   Dr = Dro + Ds*z(i);
   term2 = exp(a/Dr)*(1/Dr^2 - 2/(a*Dr) + 2/a^2);
   T(i) = To + c/Ds*(term2 - term1);
end
plot(z,T);
dTdz_zo = c/Dro^2*exp(a/Dro)
```

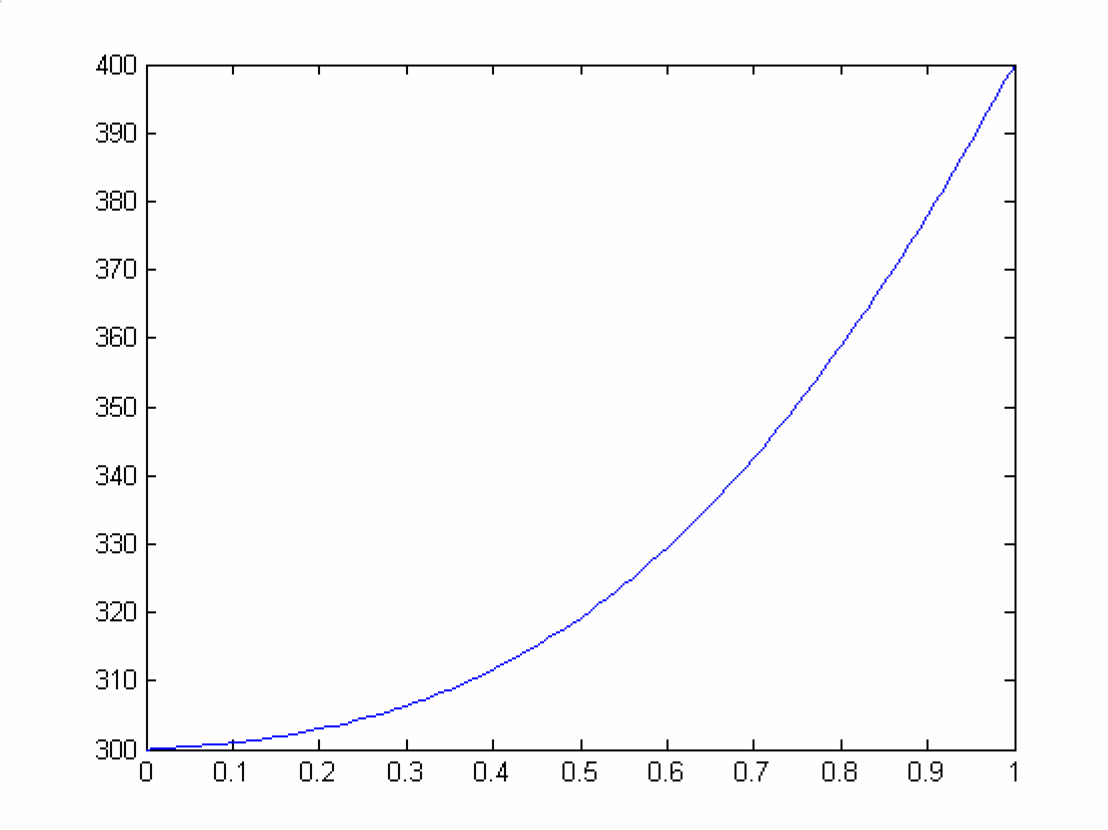
base case F=1 and kc=1



F=1 and kc=10


400
390
380
370
360
350
340
330
320
310
300
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

F=10 and kc=1


400
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320
310
300
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

For the case where the heat transfer coefficient is non-zero, the ODE is more difficult to solve analytically. The coefficients of the terms are functions of axial position, z. I am pretty sure that an analytical solution does exist, but I haven't derived it myself.

Anyway, the problem doesn't ask for an analytical derivation, only a sketch. So, once you have established what the insulated case should look like, then you have to modify that to account for heat loss. The heat loss will be greater at the wider end of the funnel so the difference between the insulated an uninsulated cases out to be greater at the wider end of the funnel. Seeing as we are on page 19 of the solutions, I am going to leave it at that.