Consider the integral equation

\[ \phi(x) = f(x) + \lambda \int_{x_0}^{x} N(x,y) \psi(y) \, dy \]

where

\[ f(x) = x^2 \]
\[ N(x,y) = x(y + 1) \]
\[ \lambda = \frac{1}{2} \]
\[ x_0 = 1 \]

(a) Is this integral equation linear or nonlinear?
(b) Is this integral equation Volterra or Fredholm?
(c) Is this integral equation of the first or second kind?
(d) Use a numerical method to find an approximate solution to \( \phi(x) \) from \( x_0 \) to \( x_f = 3 \). Use a discretization step of \( \Delta x = 1 \). You are free to solve this as you choose, as long as you state your assumptions. However, I suggest you use the trapezoidal rule to approximate the integral, although that is not mandatory. I would like to see numerical values for the solution. There is no use for calculators in this problem.
Solution:
Since the range of interest is 3-1=2 and the step size is 1, we will have n=2 intervals and n+1=3 points where the function is to be evaluated.

We write out the integral equation for each value of $x=1$, $2$, and $3$.

$x = 1: \quad \phi_1 = 1^2 + \frac{1}{2} \left[ \int_1^1 (y + 1)\phi(y)dy \right]$

$x = 2: \quad \phi_2 = 2^2 + \frac{1}{2} \left[ \int_1^2 (y + 1)\phi(y)dy \right]$

$x = 3: \quad \phi_3 = 3^2 + \frac{1}{2} \left[ \int_1^3 (y + 1)\phi(y)dy \right]$

We use the Trapezoidal rule to evaluate the integral

$$\int_{x_0}^{x_f} f(y)dy = \frac{\Delta x}{2} \left[ f(x_0) + f(x_f) + 2 \sum_{j=2}^{n} f(x_j) \right]$$

In the first equation, the integral is zero because the upper and lower limits of integration are the same. The equations become.

$x = 1: \quad \phi_1 = 1$

$x = 2: \quad \phi_2 = 4 + \frac{1}{2} \left[ (1+1)\phi_1 + (2+1)\phi_2 \right] = 4 + \phi_1 + \frac{3}{2} \phi_2$

$x = 3: \quad \phi_3 = 9 + \frac{3}{2} \left[ (1+1)\phi_1 + (3+1)\phi_3 + 2(2+1)\phi_2 \right] = 9 + \frac{3}{2} \phi_1 + \frac{9}{2} \phi_2 + 3\phi_3$

This is a set of two linear algebraic equations. There are only two unknowns because equation (1) provides the value of $\phi_1$. Now we rewrite the equations:

$x = 2: \quad -\frac{1}{2} \phi_2 = 4 + \phi_1 = 5$

$x = 3: \quad -\frac{9}{2} \phi_2 - 2 \phi_3 = 9 + \frac{3}{2} \phi_1 = 9 + \frac{3}{2} = \frac{21}{2}$

Simplify a little more for the sake of convenience

$x = 2: \quad \phi_2 = -10$
\[ x = 3 : \quad 9\phi_2 + 4 \phi_3 = -21 \]

Solving the last equation for \( \phi_3 \) yields

\[ x = 3 : \quad \phi_3 = -\frac{1}{4}(21 + 9\phi_2) = -\frac{1}{4}(-69) = \frac{69}{4} \]

So the solution is approximated by:

\[ x = 1 : \quad \phi_1 = 1 \]
\[ x = 2 : \quad \phi_2 = -10 \]
\[ x = 3 : \quad \phi_3 = \frac{69}{4} \]

This is probably a very bad approximation. We would require a much finer discretization to get a more accurate picture of the solution.