Analytical Solutions to Partial Differential Equations

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1. Categorizing PDEs

1.A. Rules of Thumb for categorizing common PDEs

What is a PDE?

A PDE is a partial differential equation. It is any equation in which there appears derivatives with respect to two different independent variables. The solution to a PDE is a function of more than one variable. Here are some examples of PDEs.

the two-dimensional Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.1}$$

the three-dimensional Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{1.2}$$

the two-dimensional Poisson equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y) \tag{1.3}$$

the one-dimensional heat equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{1.4}$$

the three-dimensional heat equation

$$\nabla \cdot [k(\nabla T)] = \frac{\partial T}{\partial t} \tag{1.5}$$

the one-dimensional wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \tag{1.6}$$

This list just shows a few examples. How we solve these PDEs varies depending on what type of PDE we have. When he were looking at ODEs, we divided our solutions methodologies according to linearity, although we could use the nonlinear numerical techniques to solve the linear problem. We then extended the methodology for a single ODE to a system of ODEs.

When we have PDEs, things get more complicated because we have two additional level of categorization. We can still have linear and nonlinear PDEs but beyond that we classify PDEs based on their order in time and their dimensionality in space. Considering first the order in time, we see that examples (1.1) to (1.3) have no time functionality in them. They are zeroth order in time. These equations are called elliptic PDEs. If we look at equations (1.4) and (1.5), they are first order in the time derivative. These equations are classified as parabolic PDEs. If we look at equation (1.6), it is second order in the time derivative. Equations of this sort are classified as hyperbolic PDEs.

- elliptic - zeroth order time derivatives
- parabolic - first order time derivatives
- hyperbolic - second order time derivatives

PDEs are also classified according to their spatial dimensionality. Thus equation (1.1) is a two-dimensional PDE because it has two spatial derivatives, in x and in y. Equation (1.2) is a three-dimensional PDE because it has three spatial derivatives, in x, y, and z. Equation (1.6) is a one-dimensional PDE because it has one spatial derivative in x.

What I have given above are rules of thumb for classify PDEs which commonly occur in engineering. What I give below is the rigorous classification for any PDE, up to second-order in the time derivative.

1.B. Rigorous categorization for any Linear PDE

Let's categorize the generic one-dimensional linear PDE which can be up to second order in the time derivative. The most general representation of this PDE is as follows:

$$A(x,t)\frac{\partial^2 U}{\partial t^2} + B(x,t)\frac{\partial^2 U}{\partial x \partial t} + C(x,t)\frac{\partial^2 U}{\partial x^2} + D(x,t)\frac{\partial U}{\partial t} + E(x,t)\frac{\partial U}{\partial x} + F(x,t)U = G(x,t) \tag{1.7}$$

When the PDE is in this form, it is difficult to determine whether it is parabolic, elliptic, or hyperbolic. However, a change of variables can be done which makes the categorization of the PDE obvious. The following results are obtained from the procedure outlined by H.F. Weinberger in “A First Course in Partial Differential Equations” (Wiley & Sons, New York, 1965, pp.41-47.)

For a given point, (x_o, t_o),the PDE is categorized as follows:

If $B^2 - 4AC > 0$ then the PDE is hyperbolic.
If $B^2 - 4AC = 0$ then the PDE is parabolic. (1.8)
If $B^2 - 4AC < 0$ then the PDE is elliptic.

This is the general classification of PDEs. As you can see, it applies only to one-dimensional, linear PDEs no greater than second order in time. This is only a subset of an infinite number of types of PDEs but, by and large, it includes much of what engineers encounter.

Let's demonstrate this with a few examples, starting with the Poisson equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x,y) \tag{1.3}$$

In this case, consider t=y, then in equation (1.8), $A = 1$, $B = 0$, $C = 1$, so that

$$B^2 - 4AC = -4 < 0$$

so we see that the Poisson equation is an elliptic PDE. If we consider the one-dimensional heat equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{1.4}$$

we have $A = 0$, $B = 0$, $C = 1$, so that

$$B^2 - 4AC = 0$$

so we see that the heat equation is a parabolic PDE. If we consider the one-dimensional wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \tag{1.6}$$

we have $A = -\frac{1}{c^2}$, $B = 0$, $C = 1$, so that

$$B^2 - 4AC = \frac{4}{c^2} > 0 \qquad \text{for all real } c$$

so we see that the wave equation is a hyperbolic PDE.

I should point out that if A, B, and C are functions of x and t, then the nature of the PDE is only valid for that point. However, if A, B, and C satisfy one of the constraints over the entire domain of interest, then the PDE can be said to be parabolic or hyperbolic or elliptic over the entire domain. If A, B, and C are constants, then the PDE will necessarily have the same character over the entire domain.

2. Single Linear Equations

In this section, I present some sample analytical solutions to single linear PDEs. The point is not to show the methodology required to obtain the solutions. The point is to show the form of the analytical solution.

A note on initial conditions and boundary conditions: Just as when we dealt with ordinary differential equations, we need 1 initial condition for each order of the maximum time derivative of the unknown function. Thus, a PDE with variable $U(x,y,z,t)$ with the largest time derivative being first order, $\frac{dU(x,y,z,t)}{dt}$, would require one initial condition, $U_o(x,y,z) = U(x,y,z,t=0)$. A PDE with variable $U(x,y,z,t)$ with the largest time derivative being second order, $\frac{d^2U(x,y,z,t)}{dt^2}$, would require two initial conditions, $U_o(x,y,z) = U(x,y,z,t=t_o)$ and $U'_o(x,y,z) = \left.\frac{dU(x,y,z,t)}{dt}\right|_{t=t_o}$. This is exactly the same as for the ODEs.

In addition to initial conditions, PDEs require boundary conditions for each spatial variable x,y,z. The number of spatial boundary conditions is not specified by the maximum order of the spatial derivative in the PDE. Rather, the number of boundary conditions is specified by the geometry of the particular problem. We shall see some examples of various boundary examples in the following problems.

The point is that in order to solve a PDE, you need a properly posed problem which includes both initial conditions and boundary conditions.

2.A Analytical Solutions to Single Linear Parabolic PDEs

We take an example from “Conduction of Heat in Solids” by H.S. Carslaw and J.C. Jaeger (Oxford Science Publications, 2nd Ed., 1959, p. 54). This text is a historical compendium of analytical solutions to various heat transfer problems. At all times, the PDE is the heat equation. However, they change the boundary conditions and internal generation terms, the coordinate system, etc. to look at different variations of the heat equation, equation (1.4).

Consider that we want to solve the heat equation for heat moving out of a slab. The slab is of width 2a running from x=-a to x=a. At time 0, the temperature of the slab is T_o and the temperature outside the slab is 0. The solution to this problem is given by:

$$T(x,t) = \frac{1}{2}T_o\left\{\mathrm{erf}\left(\frac{a-x}{2\sqrt{kt}}\right) + \mathrm{erf}\left(\frac{a+x}{2\sqrt{kt}}\right)\right\} \tag{2.1}$$

where the error function is defined as:

$$\mathrm{erf}(y) \equiv \frac{2}{\sqrt{\pi}}\int_0^y e^{-\xi^2} d\xi \tag{2.2}$$

As you can see, despite the claim to the contrary, this is hardly an analytical solution. It requires numerical calculation to evaluate the integral that has been conveniently named the error function. Plots of equation (1.9) can be found in Carslaw and Jaeger.

2.B Analytical Solutions to Single Linear Hyperbolic PDEs

We take the example of the one-dimensional wave equation, which describes the motion of a string of length L, running from x=0 to x=L

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \tag{2.3}$$

with the initial conditions:

$$U(x, t=0) = f(x) \quad \text{for} \quad 0 < x < L \tag{2.4}$$

$$\frac{dU(x, t=0)}{dt} = g(x) \quad \text{for} \quad 0 < x < L \tag{2.5}$$

and with the boundary conditions:

$$U(x=0, t) = 0 \quad \text{for} \quad t > 0 \tag{2.6}$$

$$U(x=L, t) = 0 \quad \text{for} \quad t > 0 \tag{2.7}$$

This problem with these initial and boundayr conditions is called the Dirichlet problem for th ewave equation. This problem has a solution of the form:

$$U(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi\right] \quad \text{for } t \le \frac{x}{c},\ t \le \frac{L-x}{c},\ \ t > 0 \tag{2.8}$$

This gives the solution (the position of any point of the string) at short times. The solution at all times can be determined by using some extremely cumbersome and non-intuitive reflection criteria. This solution was obtained by d’Alembert in 1747.

2.C Analytical Solutions to Single Linear Elliptic PDEs

We take the example of the two-dimensional Laplace equation, which describes the steady state (or equilibrium) distribution of temperature on a two-dimensional domain given a set of boundary conditions. (From “Theory and Problems of Partial Differential Equations”, Paul DuDhateau, David W. Zachmann, Schaum’s Outline Series, McGraw-Hill, New York, 1986, p. 97).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.9}$$

We consider the rectangular domain from x=0 to x=1 and from y=0 to y=1. We then have four boundary conditions:

$$T(x=0,y)=0 \qquad \text{for} \qquad 0<y<1 \tag{2.10}$$
$$T(x=1,y)=0 \qquad \text{for} \qquad 0<y<1 \tag{2.11}$$
$$T(x,y=0)=f_1(x) \qquad \text{for} \qquad 0<x<1 \tag{2.12}$$
$$T(x,y=1)=0 \qquad \text{for} \qquad 0<x<1 \tag{2.13}$$

The solution to this particular boundary value problem of the Laplace equation is

$$T(x,y)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right) \tag{2.14}$$

where this sort of solution is called a Fourier series. In general any attempt to expand an arbitrary function in a series of eigenfunctions is called a Fourier series. Fourier series can be used to solve elliptic, parabolic, and hyperbolic PDEs. Any solution of Laplace's equation is called a harmonic function. The constants a and b are given by

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{2.15}$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{2.16}$$

In the example above, the a_n turn out to equal zero so we arrive at a solution of the form:

$$T(x,y)=\sum_{n=1}^{\infty}c_n\sin(n\pi x)\sinh(n\pi(1-y)) \tag{2.17}$$

$$T(x,y)=\sum_{n=1}^{\infty}b_n\sin(n\pi x) \tag{2.18}$$

where

$$c_n\sinh(n\pi(1-y))=b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{2.19}$$

$$c_n = \frac{2\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh(n\pi)} \tag{2.20}$$

so

$$T(x,y) = \sum_{n=1}^{\infty} \frac{2\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\sinh(n\pi)} \sin(n\pi x)\sinh(n\pi(1-y)) \tag{2.21}$$

Now, I have a couple things to say about the supposedly analytical solution in (1.27). First, no one is going to evaluate this function analytically. There is an infinite summation. Second, the solution is only good for the very specific boundary conditions given in the example. All the same, we see the form of the solution.

3. Analytical Solutions to systems of Linear PDEs

A system of linear one-dimensional PDEs would has the form:

$$\sum_{i=1}^{n}\left[\begin{array}{l} A_{i,j}(x,t)\dfrac{\partial^2 U_j}{\partial t^2}+B_{i,j}(x,t)\dfrac{\partial^2 U_j}{\partial x\partial t}+C_{i,j}(x,t)\dfrac{\partial^2 U_j}{\partial x^2}+ \\ D_{i,j}(x,t)\dfrac{\partial U_j}{\partial t}+E_{i,j}(x,t)\dfrac{\partial U_j}{\partial x}+F_{i,j}(x,t)U_j-G_{i,j}(x,t) \end{array}\right]=0 \tag{3.1}$$

where i ranges from 1 to n.

If I am faced with a problem like this, I am going to use a numerical technique to solve it. One would be inclined to think that just as we could extend the homogenous/particular technique from a single ODE to a system of n ODEs, that we could extend some of the solutions for the single heat, wave, and Laplace equations to instances of systems of heat, wave, and Laplace equations.

Perhaps this has been done somewhere, but I have never encountered it. Therefore, I turn to a numerical solution of a form covered in the next package of lecture notes.

4. Analytical Solutions to Nonlinear PDEs

All of the analytical PDE work that we have done so far involves linear PDEs. In fact, the entire categorization as hyperbolic/parabolic/elliptic is based on the presumption of linearity.

One naturally wonders, if there exist analytical techniques to deal with nonlinear PDEs. Here are some examples of nonlinear PDEs:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{a}{T} + bT \tag{4.1}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\left(\frac{\partial T}{\partial x}\right) = 0 \tag{4.2}$$

Neither of these equations are linear because they don't satisfy the two criteria for linearity, namely,

$$L[aT] = aL[T] \tag{4.3}$$

$$L[T+U] = L[T] + L[U] \tag{4.4}$$

Well, I am an engineer and not a mathematician. That said, in my experience, I don't remember having ever been introduced to analytical techniques for solving nonlinear PDEs. I don't know that any exist. At the same time, I don't know for sure that analytical techniques for solving nonlinear PDEs don't exist. If I were to be asked to solve a nonlinear PDE, I would go straight to a numerical technique.

Likewise, if I encounter a system of nonlinear PDEs, I would, without hesitation, seek a numerical solution.

5. PDEs with more than one Spatial Dimension

Many problems have functions that vary in the x, y, and z dimensions. Modeling these problems results in multidimensional PDEs, where the unknown function can vary not only with time and with x position but also with y and z position so that in general we seek a solution to $U(x, y, z, t)$.

Some shorthand has been introduced to deal with these problems. Below, find a list of different types of derivative. After the definition, a qualitative example of the derivative is given for a fluid mechanics problem.

1. partial time derivative

$\frac{\partial \rho}{\partial t}$ = change in density at a fixed point (x,y,z) with time

We are in a canoe on a river, paddling with just enough effort so that we don't move at all. Then we look down and see $\frac{\partial \rho}{\partial t}$ at the same point. (We see different particles of fluid.)

2. total time derivative

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} \tag{5.1}$$

= change in density of the fluid while we move around with some velocity

$$\underline{v} = \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{bmatrix}^T \tag{5.2}$$

We are in a canoe on a river, paddling around, maybe cross-stream, maybe upstream, with a given velocity. Then we look down and see $\frac{\partial \rho}{\partial t}$ at different point in river and different particles of the fluid.

3. Substantial time derivative (a subset of 2.)

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}v_x + \frac{\partial \rho}{\partial y}v_y + \frac{\partial \rho}{\partial z}v_z = \frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho \tag{5.3}$$

= change in density of the fluid while we move around with the river's velocity

$$\underline{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T$$

We are in a canoe on a river, allowing ourselves to move with the river (no paddling). Then we look down and see $\frac{\partial \rho}{\partial t}$ at different point in river but for the same particle of the fluid.

4. Gradient of a scalar

$$\nabla \rho = \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \end{bmatrix}^T \tag{5.4}$$

The gradient operation on a scalar results in a 3x1 vector.

5. divergence of a vector

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{5.5}$$

The divergence operation on a vector results in a scalar. (This is the dot product of the gradient and a vector.)

6. divergence of a matrix

$$\nabla \cdot \underline{\underline{\tau}} = \begin{bmatrix} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{bmatrix} \tag{5.6}$$

The divergence operation on a vector results in a 3x1 vector. (This is the dot product of the gradient and a matrix.)

6. Laplacian of a scalar field

$$\nabla^2 \rho = \nabla \cdot \nabla \rho = \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} \tag{5.7}$$

The lapacian operation on a scalar results in a scalar. (This is the dot product of the gradient and the gradient of a scalar.)

This short-hand allows us to write the three-dimensional Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{5.8}$$

as

$$\nabla^2 T = 0 \tag{5.9}$$

or the three-dimensional heat equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \frac{d}{dy}\left(k\frac{dT}{dy}\right) + \frac{d}{dz}\left(k\frac{dT}{dz}\right) = \frac{\partial T}{\partial t} \tag{5.10}$$

as

$$\nabla \cdot [k(\nabla T)] = \frac{\partial T}{\partial t} \tag{5.11}$$

if k is a function of x, y, and z or

$$k\nabla^2 T = \frac{\partial T}{\partial t} \tag{5.12}$$

if k is a constant with respect to spatial coordinates. (k could still be a function of time.)

With simple linear problems, usually the analytical techniques that are used to solve one-dimensional problems can be extended to solve multidimensional problems. However, for anymore complicated problems, we should turn to numerical methods for solutions.

6. An application of a system of multidimensional PDEs

An example of a common system of multi-dimensional PDEs is found in fluid mechanics. In this problem, we seek to find the density and velocity of the fluid in all three directions at all points in space and time. In other words are unknowns are:
$\rho(x,y,z,t)$, $v_x(x,y,z,t)$, $v_y(x,y,z,t)$, $v_z(x,y,z,t)$.
The corresponding four PDEs are the continuity equation (a differential mass balance)

$$-\frac{D\rho}{Dt} = \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = \rho(\nabla \cdot \underline{v}) \tag{6.1}$$

and the momentum balances for the x, y, and z components:

$$-\rho\frac{D(v_x)}{Dt} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \frac{\partial p}{\partial x} - \rho g_x \tag{6.2}$$

$$-\rho\frac{D(v_y)}{Dt} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \frac{\partial p}{\partial y} - \rho g_y \tag{6.3}$$

$$-\rho\frac{D(v_z)}{Dt} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \frac{\partial p}{\partial z} - \rho g_z \qquad (6.3)$$

where $\underline{\underline{\tau}}$ is the stress matrix, $\underline{p}$ is the pressure vector, and $\underline{g}$ is the gravitational acceleration vector. In a problem $\underline{p}$ and $\underline{g}$, would be specified and $\underline{\underline{\tau}}$ would have to be given as a constituitive equation. Newton came up with one such constitutive equation, which relates the stress to the velocity gradients via a proportionality constant called the viscosity. If we limit ourselves to incompressible (constant density), Newtonian fluids, equations (5.13)-(5.16) become the continuity equation for an incompressible fluid:

$$\nabla\rho = 0 \tag{6.4}$$

and the Navier-Stokes equations:

$$\rho\frac{D(v_x)}{Dt} = \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) - \frac{\partial p}{\partial x} + \rho g_x \tag{6.5}$$

$$\rho \frac{D(v_y)}{Dt} = \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) - \frac{\partial p}{\partial y} + \rho g_y \tag{6.6}$$

$$\rho \frac{D(v_z)}{Dt} = \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \frac{\partial p}{\partial z} + \rho g_z \tag{6.7}$$

where the three equations can be expressed in vector notation as

$$\rho \frac{D(\underline{v})}{Dt} = \mu \nabla^2 \underline{v} - \nabla p + \rho \underline{g} \tag{6.8}$$

The solution of the Navier-Stokes equation for different situations with various geometries and boundary conditions seems to consume the attention of all theoretical fluid mechanicists. (This is my observation, not being a fluid mechanicist.) Once you limit yourself to incompressible, Newtonian fluids, this system of PDEs is LINEAR. All the same, various geometries and boundary conditions make the problem infinitely complex and variegated, calling invariably for numerical solution.

7. Fast Fourier Transforms

This Material is summarized from Numerical Recipes, Chapter 12, located at

http://www.ulib.org/webRoot/Books/Numerical_Recipes/bookfpdf.html

There is no point in me reworking a lesson, since a very nice lesson for FFTs is already on the web.

Go check it out.