Appendix for Lecture 28 – Linear Algebra

Included in this appendix are two sections of notes. The sections are:

A. Derivation of the explicit formula for the inverse of a 3x3 matrix *1*
B. The Basics of Eigenanalysis *5*
C. Eigenanalysis in MATLAB *9*

A. Derivation of the explicit formula for the inverse of a 3x3 matrix

For a 3x3 matrix,

$$\underline{\underline{A}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant is

$$\begin{aligned} \det\left(\underline{\underline{A}}_3\right) = {} & a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) + \\ & a_{12}\left(a_{23}a_{31} - a_{33}a_{21}\right) + \\ & a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right) \end{aligned}$$

which can be rewritten as

$$\det\left(\underline{\underline{A}}_3\right) = a_{11}\left(\det_{11}\right) + a_{12}\left(\det_{12}\right) + a_{13}\left(\det_{13}\right) = \sum_{j=1}^{3} a_{1j}\left(\det_{1j}\right)$$

where $\det_{ij}$ is the determinant of the resulting 2x2 matrix when the i^{th} row and the j^{th} column have been omitted.

STEP ONE. Write down the initial matrix augmented by the identity matrix.

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

STEP TWO. Using elementary row operations, convert $\underline{\underline{A}}$ into an identity matrix.

(1) Put a 1 in a_{11}, $\text{ROW1} = \dfrac{\text{ROW1}}{a_{11}}$

$$\left[\begin{array}{ccc|ccc} 1 & a_{12}/a_{11} & a_{13}/a_{11} & 1/a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

(2) Put zeroes in all the entries of COLUMN1 except ROW1, $\begin{array}{l} \mathrm{ROW2} = \mathrm{ROW2} - a_{21}\mathrm{ROW1} \\ \mathrm{ROW3} = \mathrm{ROW3} - a_{31}\mathrm{ROW1} \end{array}$

$$\left[\begin{array}{ccc|ccc} 1 & a_{12}/a_{11} & a_{13}/a_{11} & 1/a_{11} & 0 & 0 \\ 0 & \det_{33}/a_{11} & \det_{32}/a_{11} & -a_{21}/a_{11} & 1 & 0 \\ 0 & \det_{23}/a_{11} & \det_{22}/a_{11} & -a_{31}/a_{11} & 0 & 1 \end{array}\right]$$

(3) Put a one in a_{22}, $\mathrm{ROW2} = \dfrac{\mathrm{ROW2}}{\det_{33}/a_{11}}$

$$\left[\begin{array}{ccc|ccc} 1 & a_{12}/a_{11} & a_{13}/a_{11} & 1/a_{11} & 0 & 0 \\ 0 & 1 & \det_{32}/\det_{33} & -a_{21}/\det_{33} & a_{11}/\det_{33} & 0 \\ 0 & \det_{23}/a_{11} & \det_{22}/a_{11} & -a_{31}/a_{11} & 0 & 1 \end{array}\right]$$

(4) Put zeroes in all the entries of COLUMN2 except ROW2,

$$\mathrm{ROW1} = \mathrm{ROW1} - \frac{a_{12}}{a_{11}}\mathrm{ROW2}$$

$$\mathrm{ROW3} = \mathrm{ROW3} - \left(\det_{23}/a_{11}\right)\mathrm{ROW2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & a_{13}/a_{11} - a_{12}/a_{11}\left(\dfrac{\det_{32}}{\det_{33}}\right) & 1/a_{11} - a_{12}/a_{11}\left(\dfrac{-a_{21}}{\det_{33}}\right) & -a_{12}/\det_{33} & 0 \\ 0 & 1 & \dfrac{\det_{32}}{\det_{33}} & \dfrac{-a_{21}}{\det_{33}} & \dfrac{a_{11}}{\det_{33}} & 0 \\ 0 & 0 & \det_{22}/a_{11} - \left(\det_{23}/a_{11}\right)\left(\dfrac{\det_{32}}{\det_{33}}\right) & -a_{31}/a_{11} - \left(\det_{23}/a_{11}\right)\left(\dfrac{-a_{21}}{\det_{33}}\right) & -\dfrac{\det_{23}}{\det_{33}} & 1 \end{array}\right]$$

We will make things a little prettier if we use the relations

$$\det\left(\underline{\underline{A}}_3\right) = \frac{1}{a_{11}}\left(\det_{33}\det_{22} - \det_{32}\det_{23}\right)$$

$$\det_{13} = \frac{1}{a_{11}}\left(\det_{33} a_{31} - a_{21}\det_{23}\right)$$

$$\det_{31} = \frac{1}{a_{11}}\left(-\det_{33} a_{13} + a_{12}\det_{32}\right)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{\det_{31}}{\det_{33}} & 1/a_{11} - a_{12}/a_{11}\left(\frac{a_{21}}{\det_{33}}\right) & \frac{-a_{12}}{\det_{33}} & 0 \\ 0 & 1 & \frac{\det_{32}}{\det_{33}} & \frac{-a_{21}}{\det_{33}} & \frac{a_{11}}{\det_{33}} & 0 \\ 0 & 0 & \frac{\det\left(\underline{\underline{A}}_3\right)}{\det_{33}} & \frac{\det_{13}}{\det_{33}} & -\frac{\det_{23}}{\det_{33}} & 1 \end{array}\right]$$

(5) Put a 1 in a_{33}, $\mathrm{ROW3} = \dfrac{\mathrm{ROW3}}{\dfrac{\det\left(\underline{\underline{A}}_3\right)}{\det_{33}}}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{\det_{31}}{\det_{33}} & 1/a_{11} - a_{12}/a_{11}\left(\frac{a_{21}}{\det_{33}}\right) & \frac{-a_{12}}{\det_{33}} & 0 \\ 0 & 1 & \frac{\det_{32}}{\det_{33}} & \frac{-a_{21}}{\det_{33}} & \frac{a_{11}}{\det_{33}} & 0 \\ 0 & 0 & 1 & \frac{\det_{13}}{\det\left(\underline{\underline{A}}_3\right)} & -\frac{\det_{23}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{\det_{33}}{\det\left(\underline{\underline{A}}_3\right)} \end{array}\right]$$

(6) Put zeroes in all the entries of COLUMN3 except ROW3,

$$\mathrm{ROW1} = \mathrm{ROW1} + \frac{\det_{31}}{\det_{33}}\mathrm{ROW3}$$

$$\mathrm{ROW2} = \mathrm{ROW2} - \frac{\det_{32}}{\det_{33}}\mathrm{ROW3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a_{11}} - \frac{a_{12}}{a_{11}}\left(\frac{a_{21}}{\det_{33}}\right) + \frac{\det_{31}}{\det_{33}}\frac{\det_{13}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{-a_{12}}{\det_{33}} - \frac{\det_{31}}{\det_{33}}\frac{\det_{23}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{\det_{31}}{\det\left(\underline{\underline{A}}_3\right)} \\ 0 & 1 & 0 & \frac{-a_{21}}{\det_{33}} - \frac{\det_{32}}{\det_{33}}\frac{\det_{13}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{a_{11}}{\det_{33}} + \frac{\det_{32}}{\det_{33}}\frac{\det_{23}}{\det\left(\underline{\underline{A}}_3\right)} & -\frac{\det_{32}}{\det\left(\underline{\underline{A}}_3\right)} \\ 0 & 0 & 1 & \frac{\det_{13}}{\det\left(\underline{\underline{A}}_3\right)} & -\frac{\det_{23}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{\det_{33}}{\det\left(\underline{\underline{A}}_3\right)} \end{array}\right]$$

which can be rearranged to yield

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\det_{11}}{\det\left(\underline{\underline{A}}_3\right)} & -\frac{\det_{21}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{\det_{31}}{\det\left(\underline{\underline{A}}_3\right)} \\ 0 & 1 & 0 & -\frac{\det_{12}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{\det_{22}}{\det\left(\underline{\underline{A}}_3\right)} & -\frac{\det_{32}}{\det\left(\underline{\underline{A}}_3\right)} \\ 0 & 0 & 1 & \frac{\det_{13}}{\det\left(\underline{\underline{A}}_3\right)} & -\frac{\det_{23}}{\det\left(\underline{\underline{A}}_3\right)} & \frac{\det_{33}}{\det\left(\underline{\underline{A}}_3\right)} \end{array}\right]$$

so that we have

$$\underline{\underline{A}}_3^{-1} = \frac{1}{\det\left(\underline{\underline{A}}_3\right)}\begin{bmatrix} \det_{11} & -\det_{21} & \det_{31} \\ -\det_{12} & \det_{22} & -\det_{32} \\ \det_{13} & -\det_{23} & \det_{33} \end{bmatrix}$$

The general formula for the ijth element of $\underline{\underline{A}}_3^{-1}$ can be expressed:

$$\underline{\underline{A}}_{3\,ij}^{-1} = \frac{(-1)^{i+j}\det_{ji}}{\det\left(\underline{\underline{A}}_3\right)}$$

In terms of the elements of $\underline{\underline{A}}$, the inverse can be expressed as

$$\underline{\underline{A}}_3^{-1} = \frac{1}{\det\left(\underline{\underline{A}}_3\right)}\begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} & -a_{12}a_{33} + a_{32}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\ -a_{21}a_{33} + a_{31}a_{23} & a_{11}a_{33} - a_{31}a_{13} & -a_{11}a_{23} + a_{21}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & -a_{11}a_{32} + a_{31}a_{12} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

B. The Basics of Eigenanalysis

Eigen analysis is a part of linear algebra that is extremely important to scientists and engineers.

For an mxm square matrix, there are m eigenvalues. If the determinant of the matrix is non-zero, all of the eigenvalues are non-zero. If the determinant of the matrix is zero, at least one of the eigenvalues is zero.

To calculate the eigenvalues, $\{\lambda\}$, for an mxm matrix,

$$\underline{\underline{A}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{28.6.1}$$

first, you subtract λ from every diagonal element.

$$\underline{\underline{A}}_3 - \lambda \underline{\underline{I}}_3 = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} \tag{28.6.2}$$

Second, you set the determinant of $\underline{\underline{A}}_3 - \lambda \underline{\underline{I}}_3$ to zero,

$$\det\left(\underline{\underline{A}}_3 - \lambda \underline{\underline{I}}_3\right) = 0 \tag{28.6.3}$$

Third, you must solve this equation for λ. Equation (28.6.3) is a polynomial of mth order. It must have m roots. The m roots of equation (28.6.3) are the m eigenvalues. Equation (28.6.3) is frequently called the characteristic equation.

Each eigenvalue has associated with it an eigenvector. Thus, if there are m-eigenvectors, there are also m-eigenvalues. The mth eigenvector, $\underline{w}_m$, for the matrix in , is given by plugging the mth eigenvector, λ_m, into equation (28.6.2) for λ, and solving the equation:

$$\left(\underline{\underline{A}}_3 - \lambda_m \underline{\underline{I}}_3\right)\underline{w}_m = \underline{0} \tag{28.6.4}$$

for $\underline{w}_m$. This equation defines the eigenvectors and can be solve m times for all m eigenvalues to yield m eigenvectors.

2x2 Example:

Let's consider a 2x2 problem

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\underline{\underline{A}} - \lambda \underline{\underline{I}}_2 = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}_2\right) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12}$$

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}_2\right) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12})$$

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}_2\right) = \lambda^2 - (a_{11} + a_{22})\lambda + \det\left(\underline{\underline{A}}\right)$$

Use quadratic formula to find eigenvalues.

$$\lambda^2 - (a_{11} + a_{22})\lambda + \det\left(\underline{\underline{A}}\right) = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4\det\left(\underline{\underline{A}}\right)}}{2}$$

So we have two eigenvalues.

The eigenvalues are the same if $(a_{11} + a_{22})^2 - 4\det\left(\underline{\underline{A}}\right) = 0$

Otherwise, the eigenvalues are different.

The eigenvalues are both real if $(a_{11} + a_{22})^2 - 4\det\left(\underline{\underline{A}}\right) > 0$

The eigenvalues are complex conjugates if $(a_{11} + a_{22})^2 - 4\det\left(\underline{\underline{A}}\right) < 0$

One of the eigenvalues is zero, if $\det\left(\underline{\underline{A}}\right) = 0$.

The eigenvectors, $\underline{w}_1$ and $\underline{w}_2$ can be determined from:

$$\left(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}\right)\underline{w}_1 = \underline{0} \qquad \left(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}}\right)\underline{w}_2 = \underline{0}$$

The determinant of $\left(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}}\right)$ isalways zero, because that is how we solved for the eigenvalues. Therefore, there is always an infinite number of solutions to the eigenvectors. We know how to find one of the infinite number of roots for $\underline{\underline{A}}\underline{x} = \underline{b}$ when the determinant of $\underline{\underline{A}}$ is zero. We choose one of the elements of $\underline{x}$ arbitrarily and solve for the other elements.

$$\begin{bmatrix} a_{11}-\lambda_i & a_{12} \\ a_{21} & a_{22}-\lambda_i \end{bmatrix}\begin{bmatrix} w_{i,1} \\ w_{i,2} \end{bmatrix} = \underline{0}$$

Choose $w_{i,2} = 1$, then we solve

$$(a_{11}-\lambda_i)w_{i,1} = -a_{12}w_{i,2} \qquad \text{or} \qquad a_{21}w_{i,1} = -(a_{22}-\lambda_i)w_{i,2}$$

whichever seems reasonable.

so the eigenvectors are:

$$\underline{w}_i = \begin{bmatrix} \dfrac{-a_{12}}{a_{11}-\lambda_i} \\ 1 \end{bmatrix}$$

If we want normalized eigenvectors, then the precise choice of solution from all infinite solutions must satisfy:

$$\sqrt{\sum_{j=1}^{n} w_{i,j}^{2}} = 1$$

but what we have is

$$\sqrt{\sum_{j=1}^{n} w_{i,j}^{2}} = |\underline{w}_i|$$

which is not 1.

For the normalized eigenvectors $\underline{w'}_i$

$$\underline{w'}_i = \frac{1}{|\underline{w}_i|}\underline{w}_i = \frac{1}{\sqrt{\sum_{j=1}^{n} w_{i,j}^{2}}}\left[\begin{bmatrix} \dfrac{-a_{12}}{a_{11}-\lambda_i} \\ 1 \end{bmatrix}\right]$$

Let’s consider a 2x2 problem with numbers

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$\underline{\underline{A}} - \lambda \underline{\underline{I}}_2 = \begin{bmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}_2\right) = \lambda^2 - 5\lambda + 6$$

$$\lambda = \frac{5 \pm 1}{2} \qquad \lambda_1 = 3\,,\ \lambda_2 = 2$$

Choose $w_{i,2} = 1$, then we solve

For eigenvalue 1, we take the second equation from the matrix and we have

$$a_{21} w_{i,1} = -(a_{22} - \lambda_i) w_{i,2}$$

$$w_{i,1} = \frac{-(a_{22} - \lambda_i)}{a_{21}} w_{i,2} = \frac{-(3-2)}{1} 1 = -1$$

$$\underline{w}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For eigenvalue 2, we take the second equation from the matrix and we have

$$(a_{11} - \lambda_i) w_{i,1} = -a_{12} w_{i,2}$$

$$w_{i,1} = \frac{-a_{12}}{(a_{11} - \lambda_i)} w_{i,2} = \frac{-0}{(2-3)} 1 = 0$$

$$\underline{w}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If we want normalized eigenvectors,

$$\underline{w}'_1 = \frac{1}{|\underline{w}_1|} \underline{w}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

$$\underline{w}'_2 = \frac{1}{|\underline{w}_2|} \underline{w}_2 = \frac{1}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

An applied example of eigenanalysis is included as the last example in the hand-out on linear algebra applications.

C. Eigenanalysis in MATLAB

Given a square matrix, A, MATLAB can calculate the eigenvalues and eigenvectors using the syntax:

[W,lambda] =eig(A)

where W and lambda are both mxm matrices and where lambda is a matrix of eigenvalues in the diagonal elements and zero elsewhere, like:

$$\text{lambda}_3 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and W is a matrix where the ith column of the matrix contains the ith eigenvector

$$\underline{\underline{W}}_3 = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} = \begin{bmatrix} \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \end{bmatrix}$$

The eigenvectors MATLAB returns are normalized.