Appendix for Lecture 28 – Linear Algebra

Included in this appendix are two sections of notes. The sections are:

A. Derivation of the explicit formula for the inverse of a 3x3 matrix
B. The Basics of Eigenanalysis
C. Eigenanalysis in MATLAB

A. Derivation of the explicit formula for the inverse of a 3x3 matrix

For a 3x3 matrix,

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

the determinant is

\[
\det(A_3) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{23}a_{31} - a_{33}a_{21}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})
\]

which can be rewritten as

\[
\det(A_3) = a_{11}(\det_{11}) + a_{12}(\det_{12}) + a_{13}(\det_{13}) = \sum_{j=1}^{3} a_{1j}(\det_{1j})
\]

where \(\det_{ij}\) is the determinant of the resulting 2x2 matrix when the \(i^{th}\) row and the \(j^{th}\) column have been omitted.

STEP ONE. Write down the initial matrix augmented by the identity matrix.

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\
    a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\
    a_{31} & a_{32} & a_{33} & 0 & 0 & 1
\end{bmatrix}
\]

STEP TWO. Using elementary row operations, convert \(A\) into an identity matrix.

(1) Put a 1 in \(a_{11}\), \(ROW1 = \frac{ROW1}{a_{11}}\)
We will make things a little prettier if we use the relations
\[ \det(A_3) = \frac{1}{a_{11}} (\det A_{33} \det a_{22} - \det a_{32} \det a_{23}) \]

\[ \det_{13} = \frac{1}{a_{11}} (\det A_{33} a_{31} - a_{21} \det a_{23}) \]

\[ \det_{31} = \frac{1}{a_{11}} (-\det A_{33} a_{13} + a_{12} \det a_{32}) \]

\[
\begin{bmatrix}
1 & 0 & -\frac{\det A_{31}}{\det A_{33}} \\
0 & 1 & -\frac{\det A_{32}}{\det A_{33}} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}} (\frac{a_{21}}{\det A_{33}}) & \frac{-a_{12}}{\det A_{33}} \\
\frac{-a_{21}}{\det A_{33}} & \frac{a_{11}}{\det A_{33}} & \frac{0}{\det A_{33}} \\
\frac{-\det A_{31}}{\det A_{33}} & \frac{-\det A_{32}}{\det A_{33}} & \frac{-\det A_{33}}{\det A_{33}}
\end{bmatrix}
\]

(5) Put a 1 in \( a_{33} \), \( \text{ROW3} = \frac{\text{ROW3}}{\det A_{33}} \)

\[
\begin{bmatrix}
1 & 0 & -\frac{\det A_{31}}{\det A_{33}} \\
0 & 1 & -\frac{\det A_{32}}{\det A_{33}} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}} (\frac{a_{21}}{\det A_{33}}) & \frac{-a_{12}}{\det A_{33}} \\
\frac{-a_{21}}{\det A_{33}} & \frac{a_{11}}{\det A_{33}} & \frac{0}{\det A_{33}} \\
\frac{-\det A_{31}}{\det A_{33}} & \frac{-\det A_{32}}{\det A_{33}} & \frac{-\det A_{33}}{\det A_{33}}
\end{bmatrix}
\]

(6) Put zeroes in all the entries of COLUMN3 except ROW3,

\[
\text{ROW1} = \text{ROW1} + \frac{\det A_{31}}{\det A_{33}} \text{ROW3}
\]

\[
\text{ROW2} = \text{ROW2} - \frac{\det A_{32}}{\det A_{33}} \text{ROW3}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1/a_{11} & -a_{12}/a_{11} & a_{21}/a_{11} \det(A_3) \ 
-\frac{a_{21}}{\det(A_3)} & \frac{\det_1}{\det(A_3)} & \frac{\det_2}{\det(A_3)} \ 
\frac{\det_3}{\det(A_3)} & -\frac{a_{12}}{\det(A_3)} & \frac{\det_3}{\det(A_3)} \ 
\end{bmatrix}
\begin{bmatrix}
\det(A_3) & \det(A_3) & \det(A_3) \\
\det(A_3) & \det(A_3) & \det(A_3) \\
\det(A_3) & \det(A_3) & \det(A_3) \\
\end{bmatrix}
\]

which can be rearranged to yield

\[
\begin{bmatrix}
\det_1 & \det_1 & \det_1 \\
\frac{\det_1}{\det(A_3)} & \frac{\det_2}{\det(A_3)} & \frac{\det_3}{\det(A_3)} \\
\frac{\det_3}{\det(A_3)} & -\frac{\det_2}{\det(A_3)} & -\frac{\det_3}{\det(A_3)} \\
\end{bmatrix}
\begin{bmatrix}
\det(A_3) & \det(A_3) & \det(A_3) \\
\det(A_3) & \det(A_3) & \det(A_3) \\
\det(A_3) & \det(A_3) & \det(A_3) \\
\end{bmatrix}
\]

so that we have

\[
A_3^{-1} = \frac{1}{\det(A_3)}
\begin{bmatrix}
\det_1 & -\det_2 & \det_3 \\
-\det_1 & \det_2 & -\det_3 \\
\det_1 & -\det_2 & \det_3 \\
\end{bmatrix}
\]

The general formula for the $ij$th element of $A_3^{-1}$ can be expressed:

\[
A_3^{-1}_{ij} = \frac{(-1)^{i+j}\det_{ji}}{\det(A_3)}
\]

In terms of the elements of $A$, the inverse can be expressed as

\[
A_3^{-1} = \frac{1}{\det(A_3)}
\begin{bmatrix}
a_{22}a_{33} - a_{32}a_{23} & -a_{12}a_{33} + a_{32}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\
-a_{21}a_{33} + a_{31}a_{23} & a_{11}a_{33} - a_{31}a_{13} & -a_{11}a_{23} + a_{21}a_{13} \\
a_{21}a_{32} - a_{31}a_{22} & -a_{11}a_{32} + a_{31}a_{12} & a_{11}a_{22} - a_{21}a_{12} \\
\end{bmatrix}
\]
B. The Basics of Eigenanalysis

Eigen analysis is a part of linear algebra that is extremely important to scientists and engineers.

For an \( m \times m \) square matrix, there are \( m \) eigenvalues. If the determinant of the matrix is non-zero, all of the eigenvalues are non-zero. If the determinant of the matrix is zero, at least one of the eigenvalues is zero.

To calculate the eigenvalues, \( \{ \lambda \} \), for an \( m \times m \) matrix,

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

first, you subtract \( \lambda \) from every diagonal element.

\[
A - \lambda I = \begin{bmatrix}
a_{11} - \lambda & a_{12} & a_{13} \\
a_{21} & a_{22} - \lambda & a_{23} \\
a_{31} & a_{32} & a_{33} - \lambda
\end{bmatrix}
\]

(28.6.2)

Second, you set the determinant of \( A - \lambda I \) to zero,

\[
\det(A - \lambda I) = 0
\]

(28.6.3)

Third, you must solve this equation for \( \lambda \). Equation (28.6.3) is a polynomial of mth order. It must have m roots. The m roots of equation (28.6.3) are the m eigenvalues. Equation (28.6.3) is frequently called the characteristic equation.

Each eigenvalue has associated with it an eigenvector. Thus, if there are m-eigenvectors, there are also m-eigenvalues. The \( m^{th} \) eigenvector, \( \mathbf{w}_m \), for the matrix in , is given by plugging the \( m^{th} \) eigenvector, \( \lambda_m \), into equation (28.6.2) for \( \lambda \), and solving the equation:

\[
(A - \lambda_m I) \mathbf{w}_m = 0
\]

(28.6.4)

for \( \mathbf{w}_m \). This equation defines the eigenvectors and can be solve m times for all m eigenvalues to yield m eigenvectors.

2x2 Example:

Let’s consider a 2x2 problem

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]
\[
\begin{array}{c}
\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix}
a_{11} - \lambda & a_{12} \\
a_{21} & a_{22} - \lambda
\end{bmatrix}
\end{array}
\]

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12}
\]

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12})
\]

\[
\det(\mathbf{A} - \lambda \mathbf{I}) = \lambda^2 - (a_{11} + a_{22})\lambda + \det(\mathbf{A})
\]

Use quadratic formula to find eigenvalues.

\[
\lambda^2 - (a_{11} + a_{22})\lambda + \det(\mathbf{A}) = 0
\]

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4\det(\mathbf{A})}}{2}
\]

So we have two eigenvalues.

The eigenvalues are the same if \((a_{11} + a_{22})^2 - 4\det(\mathbf{A}) = 0\)

Otherwise, the eigenvalues are different.

The eigenvalues are both real if \((a_{11} + a_{22})^2 - 4\det(\mathbf{A}) > 0\)

The eigenvalues are complex conjugates if \((a_{11} + a_{22})^2 - 4\det(\mathbf{A}) < 0\)

One of the eigenvalues is zero, if \(\det(\mathbf{A}) = 0\).

The eigenvectors, \(\mathbf{w}_1\) and \(\mathbf{w}_2\) can be determined from:

\[
(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{w}_1 = 0 \quad (\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{w}_2 = 0
\]

The determinant of \((\mathbf{A} - \lambda \mathbf{I})\) is always zero, because that is how we solved for the eigenvalues. Therefore, there is always an infinite number of solutions to the eigenvectors. We know how to find one of the infinite number of roots for \(\mathbf{A}\mathbf{x} = \mathbf{b}\) when the determinant of \(\mathbf{A}\) is zero. We choose one of the elements of \(\mathbf{x}\) arbitrarily and solve for the other elements.
Choose \( w_{i,2} = 1 \), then we solve

\[
(a_{11} - \lambda_i)w_{i,1} = -a_{12}w_{i,2} \quad \text{or} \quad a_{21}w_{i,1} = -(a_{22} - \lambda_i)w_{i,2}
\]

whichever seems reasonable.

so the eigenvectors are:

\[
w_i = \begin{bmatrix}
a_{12} \\
a_{11} - \lambda_i \\
1
\end{bmatrix}
\]

If we want normalized eigenvectors, then the precise choice of solution from all infinite solutions must satisfy:

\[
\sqrt{\sum_{j=1}^{n} w_{i,j}^2} = 1
\]

but what we have is

\[
\sqrt{\sum_{j=1}^{n} w_{i,j}^2} = |w_i|
\]

which is not 1.

For the normalized eigenvectors \( w'_i \)

\[
w'_i = \frac{1}{|w_i|} w_i = \frac{1}{\sqrt{\sum_{j=1}^{n} w_{i,j}^2}} \begin{bmatrix}
-a_{12} \\
a_{11} - \lambda_i \\
1
\end{bmatrix}
\]

Let’s consider a 2x2 problem with numbers

\[
A = \begin{bmatrix}
2 & 0 \\
1 & 3
\end{bmatrix}
\]

\[
A - \lambda I_2 = \begin{bmatrix}
2 - \lambda & 0 \\
1 & 3 - \lambda
\end{bmatrix}
\]
\[
\det(A - \lambda I_2) = \lambda^2 - 5\lambda + 6 \\
\lambda = \frac{5 \pm 1}{2} \quad \lambda_1 = 3, \lambda_2 = 2
\]

Choose \( w_{i2} = 1 \), then we solve

For eigenvalue 1, we take the second equation from the matrix and we have

\[
a_{21}w_{i1} = -(a_{22} - \lambda_i)w_{i2} \\
w_{i1} = -\frac{(a_{22} - \lambda_i)}{a_{21}}w_{i2} = -\frac{(3 - 2)}{1} = -1
\]

\[
w_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

For eigenvalue 2, we take the second equation from the matrix and we have

\[
(a_{11} - \lambda_i)w_{i1} = -a_{12}w_{i2} \\
w_{i1} = -\frac{a_{12}}{(a_{11} - \lambda_i)}w_{i2} = -\frac{0}{(2 - 3)} = 0
\]

\[
w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

If we want normalized eigenvectors,

\[
w'_1 = \frac{1}{\|w_1\|}w_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}
\]

\[
w'_2 = \frac{1}{\|w_2\|}w_2 = \frac{1}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

An applied example of eigenanalysis is included as the last example in the hand-out on linear algebra applications.
C. Eigenanalysis in MATLAB

Given a square matrix, A, MATLAB can calculate the eigenvalues and eigenvectors using the syntax:

\[ [W, \lambda] = \text{eig}(A) \]

where \( W \) and \( \lambda \) are both \( mxm \) matrices and where \( \lambda \) is a matrix of eigenvalues in the diagonal elements and zero elsewhere, like:

\[ \lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \]

and \( W \) is a matrix where the \( i \)th column of the matrix contains the \( i \)th eigenvector

\[ W = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \]

The eigenvectors MATLAB returns are normalized.