Lecture 28-30 - Solution of a System of Linear Equation

28.1 Why is it important to be able to solve a system of linear equations?

Many of the phenomena observed in science and engineering can be described by linear equations. The vibrational modes of a crystalline material, and the equilibrium of a multicomponent-multi-reaction vessel are just two examples of problems that give rise to systems of linear, algebraic equations.

Of course, there are many systems that are intrinsically nonlinear. For example, material and energy balances are both nonlinear equations. However, we begin a study of the solution of equations by focussing on linear equations for several reasons

- linear equations are easier to solve than nonlinear equations
- more can be known about the existence and uniqueness of solutions for linear equations than can be known for nonlinear equations
- by observing the behavior of solution techniques for linear equations, we may get an idea about how solution methods may work or fail for nonlinear equations
- there is a tremendous amount of theory of linear algebra, which provides insight into the solution of special systems: symmetric matrices, sparse matrices, banded matrices, repeated solutions, etc.

28.2 Linear Algebraic Equations

An equation is linear if the unknowns in the equation appears as sums or differences. If their is multiplication, division, or any transcendental function of the unknown, then the equation is nonlinear.

example of a linear equation:

$$2x + 5 = 0$$

example of nonlinear equations:

$$2x^2 + 5 = 0$$
$$2\log(x) + 5 = 0$$

A system of equations is linear if all the unknowns appear only as sums or differences.

example of a system of 2 linear equations:

$$2x_1 - 3x_2 + 5 = 0$$
$$2x_1 - 3x_2 + 5 = 0$$

example of a system of 2 nonlinear equations:

$$2\sin(x_1) - 3x_2 + 5 = 0$$
$$2x_1x_2 + 5 = 0$$

If any one equation in the system is nonlinear, then the system is considered nonlinear.

28.3 Converting systems of linear equations into matrix notation

We will find that it conserves space to write systems of equations in matrix notation. For the general system of n linear equations with n unknowns, $x_1, x_2, x_3 ... x_n$, we can write:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots + a_{1,n-1}x_{n-1} + a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots + a_{2,n-1}x_{n-1} + a_{2,n}x_n = b_2$$
$$\ldots$$
$$a_{n,1}x_1 + a_{n,2}x_2 + a_{n,3}x_3 + \ldots + a_{n,n-1}x_{n-1} + a_{n,n}x_n = b_n$$

The a's and b's are constants. Each a has two subscripts. The first subscript on a indicates the equation it appears in. The second subscript on a indicates the variable it appears in front of. Each b has one subscript, indicating which equation it appears in.

This system of linear algebraic equations can be written in matrix notation as:

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

where $\underline{\underline{A}}$ is a matrix of size nxn, $\underline{x}$ is a vector of size nx1 and $\underline{b}$ is a vector of size nx1. Specifically,

$$\underline{\underline{A}} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}, \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

An example. The system of 2 equations and 2 unknowns,

$$2x_1 + 4x_2 = 0$$
$$-3x_1 + 9x_2 = 8$$

can be written as:

$$\begin{bmatrix} 2 & 4 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

Any system of linear algebraic equations can be converted to matrix form.

28.4 Extending ordinary algebra to linear algebra

Say we have the system of 2 linear equations:

$$a_{1,1}x_1 + a_{1,2}x_2 = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 = b_2$$

If we want to solve these equations using traditional algebraic techniques. We isolate x_1 in equation 1, and substitute it into equation 2. Then solve for x_2 in equation 2.

$$x_1 = \frac{b_1 - a_{1,2}x_2}{a_{1,1}} \qquad \text{(1) (solve equation 1 for } x_1\text{)}$$

$$a_{2,1}\frac{b_1 - a_{1,2}x_2}{a_{1,1}} + a_{2,2}x_2 = b_2 \qquad \text{(2) (substitute } x_1 \text{ into equation 2)}$$

$$a_{2,1}\frac{b_1 - a_{1,2}x_2}{a_{1,1}} + a_{2,2}x_2 = b_2 \qquad \text{(3) (solve for } x_2\text{)}$$

$$x_2 = \frac{b_2 - \frac{a_{2,1}b_1}{a_{1,1}}}{\left(a_{2,2} - \frac{a_{2,1}a_{1,2}}{a_{1,1}}\right)} = \frac{a_{1,1}b_2 - a_{2,1}b_1}{\left(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}\right)} \qquad \text{(4) (solve for } x_2\text{)}$$

$$x_1 = \frac{b_1}{a_{1,1}} - \frac{a_{1,2}}{a_{1,1}}\frac{a_{1,1}b_2 - a_{2,1}b_1}{\left(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}\right)} \qquad \text{(5) (solve (1) for } x_1\text{)}$$

The problem with this technique of substitution is that it takes a long time as the number of equations increase. We need a quicker, more methodical approach to solving systems of linear algebraic equations.

We can write our original equations in matrix notation as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

We define an inverse matrix, $\underline{\underline{A}}^{-1}$, which is the same size as the matrix $\underline{\underline{A}}$, namely 2x2 in this case, which has the property:

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$$

where $\underline{\underline{I}}$ is called the identity matrix and is defined as:

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The size of $\underline{I}$ is the same as the size of $\underline{\underline{A}}$. The most important property of the identity matrix is that multiplying a matrix or vector by $\underline{I}$ yields the original matrix or vector:

$$\underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}} \qquad\qquad \underline{\underline{I}}\,\underline{b} = \underline{b}$$

If we had this magical creature called an inverse then we could solve the system of equations easily:

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$\underline{\underline{A}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

$$\underline{\underline{I}}\,\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

$$\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$$

Thus we would have the vector of solutions, $\underline{x}$. So…we need to know how and when we can compute inverses.

28.5 Determinants and inverses

The inverse only exists for square matrices. (That is the dimensions of the matrix are n by n.)

The first time we calculate an inverse, we will use what is called Naïve Gauss Elimination (NGE). In NGE, we use three elementary row operations. These elementary row operations are

$\text{row }1 = c \cdot \text{row }1$ (multiplication of a row by a constant)

$\text{row }2 = a \cdot \text{row }1 + b \cdot \text{row}2$ (replacement of a row with a linear combination of that row and another row)

$\text{row }2 \leftrightarrow \text{row }1$ (swapping rows)

Under certain conditions, NGE allows us to find the inverse. For a 2x2 matrix, the procedure for finding the inverse is given below:

STEP ONE. Write down the initial matrix augmented by the identity matrix.

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right]$$

STEP TWO. Using elementary row operations, convert $\underline{\underline{A}}$ into an identity matrix.

(1) Put a 1 in a_{11}, $\text{ROW1} = \dfrac{\text{ROW1}}{a_{11}}$

$$\left[\begin{array}{cc|cc} 1 & a_{12}/a_{11} & 1/a_{11} & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right]$$

(2) Put zeroes in all the entries of COLUMN1 except ROW1, $ROW2 = ROW2 - a_{21}ROW1$

$$\left[\begin{array}{cc|cc} 1 & a_{12}/a_{11} & 1/a_{11} & 0 \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & -a_{21}/a_{11} & 1 \end{array}\right]$$

(3) Put a one in a_{22}, $ROW2 = \dfrac{ROW2}{a_{22} - a_{21}a_{12}/a_{11}}$

$$\left[\begin{array}{cc|cc} 1 & a_{12}/a_{11} & 1/a_{11} & 0 \\ 0 & 1 & \dfrac{-a_{21}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}} & \dfrac{1}{a_{22} - a_{21}a_{12}/a_{11}} \end{array}\right]$$

(4) Put zeroes in all the entries of COLUMN1 except ROW2, $ROW1 = ROW1 - \dfrac{a_{12}}{a_{11}} ROW2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/a_{11} - \dfrac{a_{12}}{a_{11}}\left(\dfrac{-a_{21}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}}\right) & -\dfrac{a_{12}}{a_{11}}\left(\dfrac{1}{a_{22} - a_{21}a_{12}/a_{11}}\right) \\ 0 & 1 & \dfrac{-a_{21}/a_{11}}{a_{22} - a_{21}a_{12}/a_{11}} & \dfrac{1}{a_{22} - a_{21}a_{12}/a_{11}} \end{array}\right]$$

which can be simplified as:

$$\left[\begin{array}{cc|cc} 1 & 0 & \dfrac{a_{11}}{a_{11}a_{22} - a_{21}a_{12}} & \dfrac{-a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \\ 0 & 1 & \dfrac{-a_{21}}{a_{11}a_{22} - a_{21}a_{12}} & \dfrac{a_{11}}{a_{11}a_{22} - a_{21}a_{12}} \end{array}\right]$$

Here we have converted the matrix on the left hand side to the identity matrix. As a result the identity matrix that was originally on the right-hand side is now the inverse of A, $\underline{\underline{A}}^{-1}$.

This inverse can be rewritten as

$$\underline{\underline{A}}^{-1} = \frac{a_{11}}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{28.21}$$

where the determinant of $\underline{\underline{A}}$ (a 2x2 matrix), $\det(\underline{\underline{A}})$, is given to be:

$$\det(\underline{\underline{A}}) = a_{11}a_{22} - a_{21}a_{12}$$

We can learn several things about the inverse from this demonstration. The most important thing is:

If the determinant is zero, the inverse does not exist (because we divide by the determinant to obtain the inverse.) A matrix with a determinant of zero is called *singular*. A matrix with a non-zero determinant is called *non-singular*.

Never calculate an inverse until you have first shown that the determinant is not zero.

We can check that this is the correct inverse by substituting $\underline{\underline{A}}^{-1}$ into

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\underline{\underline{A}}^{-1} = \underline{\underline{I}}$$

and verifying that we obtain the identity matrix.

For a 3x3 matrix,

$$\underline{\underline{A}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{28.18}$$

the determinant is

$$\begin{aligned} \det(\underline{\underline{A}}_3) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + \\ a_{12}(a_{23}a_{31} - a_{33}a_{21}) + \\ a_{13}(a_{21}a_{32} - a_{31}a_{22}) \end{aligned} \tag{28.19}$$

NGE can be applied to any mxm matrix with a non-zero determinant. A clean formula like equation (28.20) is not available for m > 2. Even for m = 3, the analogous formula tends toward the gruesome. (See the appendix for the complete derivation.)

$$\underline{\underline{A}}_3^{-1} = \frac{1}{\det\left(\underline{\underline{A}}_3\right)} \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} & -a_{12}a_{33} + a_{32}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\ -a_{21}a_{33} + a_{31}a_{23} & a_{11}a_{33} - a_{31}a_{13} & -a_{11}a_{23} + a_{21}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & -a_{11}a_{32} + a_{31}a_{12} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix} \quad (28.24)$$

For larger matrices, we generally turn to computers to calculate the determinant and inverse for us.

28.6 Rank and Row Echelon Form

We need to introduce a couple additional quantities before we get around to using the inverse to solve $\underline{\underline{A}}\underline{x} = \underline{b}$.

We say that a matrix is in *Row Echelon Form* when all elements below the diagonal are zero. This notation is also called an upper triangular matrix. Starting with the matrix $\underline{\underline{A}}_3$, we perform elementary row operations on the matrix until the we zero out the required elements.

$$\underline{\underline{A}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{ROW2} = \text{ROW1} - \frac{a_{11}}{a_{21}}\text{ROW2}$$

$$\text{ROW3} = \text{ROW1} - \frac{a_{11}}{a_{31}}\text{ROW3}$$

$$\underline{\underline{U}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{12} - \frac{a_{11}}{a_{21}}a_{22} & a_{13} - \frac{a_{11}}{a_{21}}a_{23} \\ 0 & a_{12} - \frac{a_{11}}{a_{31}}a_{32} & a_{13} - \frac{a_{11}}{a_{31}}a_{33} \end{bmatrix}$$

$$\text{ROW3} = \text{ROW2} - \frac{a_{12} - \frac{a_{11}}{a_{21}}a_{22}}{a_{12} - \frac{a_{11}}{a_{31}}a_{32}}\text{ROW3}$$

Simplification yields:

$$\underline{\underline{U}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{12} - \frac{a_{11}}{a_{21}}a_{22} & a_{13} - \frac{a_{11}}{a_{21}}a_{23} \\ 0 & 0 & \det\left(\underline{\underline{A}}_3\right) \end{bmatrix}$$

The *rank* of A is the number of non-zero rows in the matrix when it is put in row-echelon form. The *rank* of A is also the number of independent equations in A.

If the determinant of the matrix is non-zero,we see that the rank of an nxn matrix is n.

$$\underline{\underline{U}} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

If the determinant of an nxn matrix is zero, then the $\text{rank}(\underline{\underline{A}}_n)$ is less than n.

$$\underline{\underline{U}} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

Non-square matrices can also be put in row echelon form. Consider an nx(n+1) matrix of the form:

$$\underline{\underline{C}} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Again, by performing elementary row operations, we reduce this matrix to an upper triangular form,

$$\underline{\underline{U}} = \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & v_1 \\ 0 & u_{22} & u_{23} & v_2 \\ 0 & 0 & u_{33} & v_3 \end{array}\right]$$

The rank of this matrix is still defined as the number of non-zero rows in the row echelon form of the matrix. The rank of the following matrix is three.

$$\underline{\underline{U}} = \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & v_1 \\ 0 & u_{22} & u_{23} & v_2 \\ 0 & 0 & 0 & v_3 \end{array}\right]$$

The rank of the following matrix is two.

$$\underline{\underline{U}} = \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & v_1 \\ 0 & u_{22} & u_{23} & v_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If a matrix has some non-zero rows, then it means that some of the equations were linearly independent. Consider the matrix below.

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{11} + ka_{21} & ca_{12} + ka_{22} & ca_{13} + ka_{23} \end{bmatrix}$$

We can clearly see that Row 3 = c*Row 1 + k*Row 2. Row 3 is a linear combinations of Rows 1 and 2. If we perform elementary row operations on $\underline{\underline{A}}$ to put it in row echelon form, then we will find that there are two non-zero rows. Thus the rank is 2, the number of independent equations.

At this point we can identify some logically equivalent statements about an nxn matrix, $\underline{\underline{A}}$. If any one of these statements is true, all the others are true.

• If and only if $\det(\underline{\underline{A}}) \neq 0$ • then inverse exists • then $\underline{\underline{A}}$ is non-singular • then $\text{rank}(\underline{\underline{A}}) = n$ • then there are no zero rows in the row echelon form of $\underline{\underline{A}}$ • then $\underline{\underline{A}}\underline{x} = \underline{b}$ has one, unique solution • all eigenvalues of $\underline{\underline{A}}$ are non-zero	• If and only if $\det(\underline{\underline{A}}) = 0$ • then inverse does not exist • then $\underline{\underline{A}}$ is singular • then $\text{rank}(\underline{\underline{A}}) < n$ • then there is at least one zero row in the row echelon form of $\underline{\underline{A}}$ • then $\underline{\underline{A}}\underline{x} = \underline{b}$ has either no solution or infinite solutions • at least one eigenvalue of $\underline{\underline{A}}$ is zero

28.7 Existence and Uniqueness of Solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$

We now have all the tools we need to solve $\underline{\underline{A}}\underline{x} = \underline{b}$. Before we work any examples, we need to know beforehand how many solutions we can obtain for a system. In dealing with linear equations, we only have three choices for the number of solutions. We either have 0, 1, or an infinite number of solutions.

No Solutions:

$$\text{rank}(\underline{\underline{A}}) < n \text{ and } \text{rank}(\underline{\underline{A}}) < \text{rank}(\underline{\underline{A}}|\underline{b})$$

One Solution:

$$\text{rank}(\underline{\underline{A}}) = \text{rank}(\underline{\underline{A}}|\underline{b}) = n$$

Infinite Solutions:

$$\text{rank}(\underline{\underline{A}}) = \text{rank}(\underline{\underline{A}}|\underline{b}) < n$$

Consider each case.

When $\text{rank}(\underline{\underline{A}}|\underline{b}) > \text{rank}(\underline{\underline{A}})$, your system is over-specified. There are no solutions to your problem.

When $\text{rank}(\underline{\underline{A}}|\underline{b}) = \text{rank}(\underline{\underline{A}}) = n$, you have a properly specified system with n equations and n unknowns and you have one, unique solution.

When $\text{rank}(\underline{\underline{A}}|\underline{b}) = \text{rank}(\underline{\underline{A}}) < n$, then you have less equations than unknowns. You can pick $n - \text{rank}(\underline{\underline{A}})$ unknowns arbitrarily then solve for the rest. Therefore you have an infinite number of solutions.

We will work one example of each case below.

Example: One Solution to $\underline{\underline{A}}\underline{x} = \underline{b}$

Let's find

(a) the determinant of $\underline{\underline{A}}$

(b) the inverse of $\underline{\underline{A}}$

(c) the solution of $\underline{\underline{A}}\underline{x} = \underline{b}_1$

(d) the solution of $\underline{\underline{A}}\underline{x} = \underline{b}_2$

where

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \underline{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) The determinant of $\underline{\underline{A}}$ is (by equation 28.19) $\det(\underline{\underline{A}}) = -1$

(b) Because the determinant is non-zero, we know there will be an inverse. Let's find it.

STEP ONE. Write down the initial matrix augmented by the identity matrix.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

STEP TWO. Using elementary row operations, convert $\underline{\underline{A}}$ into an identity matrix.

(1) Put a 1 in a_{11}, $ROW1 = \frac{ROW1}{a_{11}} = \frac{ROW1}{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

(2) Put zeroes in all the entries of COLUMN1 except ROW1,

$ROW2 = ROW2 - a_{21}ROW1 = ROW2 - ROW1$

$ROW3 = ROW3 - a_{31}ROW1 = ROW3 - ROW1$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & -1/2 & -1/2 & 0 & 1 \end{array}\right]$$

(3) Put a 1 in a_{22}, $ROW2 = \frac{ROW2}{a_{22}} = \frac{ROW2}{3/2}$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1/3 & -1/3 & 2/3 & 0 \\ 0 & 1/2 & -1/2 & -1/2 & 0 & 1 \end{array}\right]$$

(4) Put zeroes in all the entries of COLUMN2 except ROW2,

$ROW1 = ROW1 - a_{12}ROW2 = ROW1 - 1/2 * ROW2$

$ROW3 = ROW3 - a_{32}ROW2 = ROW3 - 1/2 * ROW2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & -1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & -1/3 & -1/3 & -1/3 & 1 \end{array}\right]$$

(5) Put a 1 in a_{33}, $ROW3 = \frac{ROW3}{a_{33}} = \frac{ROW3}{-1/3}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & -1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{array}\right]$$

(6) Put zeroes in all the entries of COLUMN3 except ROW3,

$$ROW1 = ROW1 - a_{13}ROW3 = ROW1 - 5/3 * ROW3$$

$$ROW2 = ROW2 - a_{23}ROW3 = ROW2 + 1/3 * ROW3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 5 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -3 \end{array}\right]$$

so

$$\underline{\underline{A}}^{-1} = \begin{bmatrix} -1 & -2 & 5 \\ 0 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

(c) The solution to $\underline{\underline{A}}\underline{x} = \underline{b}_1$ is $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}_1$

$$\underline{x} = \begin{bmatrix} -1 & -2 & 5 \\ 0 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

(d) The solution to $\underline{\underline{A}}\underline{x} = \underline{b}_2$ is $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}_2$

$$\underline{x} = \begin{bmatrix} -1 & -2 & 5 \\ 0 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -4 \end{bmatrix}$$

We see that we only need to calculate the inverse once to solve both $\underline{\underline{A}}\underline{x} = \underline{b}_1$ and $\underline{\underline{A}}\underline{x} = \underline{b}_2$. That's nice because finding the inverse is a lot harder than solving the equation once the inverse is known.

Example: No Solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 3 & 4 \end{bmatrix} \quad \underline{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\det(\underline{\underline{A}}) = 0$$

The determinant is zero. No inverse exists. To determine if we have no solution or infinite solutions find the ranks of A and A|b. In row echelon form, A becomes:

$$\underline{\underline{U}}_A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the $\text{rank}(\underline{\underline{A}}) = 2$

In row echelon form, A|b becomes:

$$\underline{\underline{U}}_{A|b} = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

so the $\text{rank}(\underline{\underline{A\,|\,b}}) = 3$.

Since $\text{rank}(\underline{\underline{A\,|\,b}}) > \text{rank}(\underline{\underline{A}})$, there are no solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$.

Example: Infinite Solutions to $\underline{\underline{A}}\underline{x} = \underline{b}$

Consider the same matrix, $\underline{\underline{A}}$, as was used in the previous example. The determinant is zero and the rank is 2. now consider a different b vector.

$$\underline{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Reduce the $\underline{\underline{A}}|\underline{b}$ matrix to row echelon form.

$$\underline{\underline{U}}_{A|b} = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this case, $\text{rank}(\underline{\underline{A\,|\,b}}) = 2$.

Since $\text{rank}(\underline{\underline{A}}) = \text{rank}(\underline{\underline{A}}|\underline{b}) = 2 < n = 3$, there are infinite solutions.

We can find one example of the infinite solutions by following a standard procedure. First, we arbitrarily select $n - \text{rank}(\underline{\underline{A}})$ variables. In this case we can select one variable. Let's make $x_3 = 0$.

Then substitute that value into the row echelon form of $\underline{\underline{A}}|\underline{b}$ and solve the resulting system of $\text{rank}(\underline{\underline{A}})$ equations.

$$\underline{\underline{U}}_{A|b} = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

When $x_3 = 0$

$$\underline{\underline{U}}_{A|b} = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now solve a new $\underline{\underline{A}}\underline{x} = \underline{b}$ problem where A and b come from the non-zero parts of $\underline{\underline{U}}_{A|b}$

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This problem will always have an inverse.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

So one example of the infinite solutions is

$$\underline{x} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \end{bmatrix}$$

28.8 Eigenvalues and Eigenvectors

See Linear Algebra Appendix

28.9 Example Applications

See Linear Algebra Applications Packet

Summary of basic MATLAB commands for Linear Algebra

<table>
<tr><td colspan="2"><u>Entering a matrix</u>

<code>A=[a11,12;a21,a22]</code>

(commas separate elements in a row, semicolons separate rows)
(easiest for direct data entry)

<code>A=[a11   a12</code>
<code>a21   a22]</code>

(tabs separate elements in a row, returns separate rows)
(useful for copying data from a table in Word or Excel)</td></tr>
<tr><td><u>Entering a column vector</u>

<code>b=[b1;b2;b3]</code>

(an nx1 vector)</td><td><u>Entering a row vector</u>

<code>b=[b1,b2,b3]</code>

(a 1xn vector)</td></tr>
<tr><td><u>determinant of a matrix</u>

<code>det(A)</code>

(scalar)</td><td><u>rank of a matrix</u>

<code>rank(A)</code>

(scalar)</td></tr>
<tr><td><u>inverse of an nxn matrix</u>

<code>inv(A)</code>

(nxn matrix)</td><td><u>transpose of an nxm matrix or an nx1 vector</u>

<code>A=A’</code>

(mxn matrix or 1xn vector)</td></tr>
<tr><td><u>solution of Ax=b</u>

<code>x=A\b</code> or <code>x=inv(A)*b</code>

(nx1 vector)</td><td><u>reduced row echelon form of an nxn matrix</u>

<code>rref(A)</code>

(nxn matrix)</td></tr>
<tr><td colspan="2"><u>eigenvalues and eigenvector of an nxn matrix</u>

<code>[w,lambda]=eig(A)</code>

(w is an nxn matrix where each column is an eigenvector,
lambda is a nxn matrix where each diagonal element is an eigenvalue, off-diagonals are zero).</td></tr>
</table>