Lectures 4 - Random Variables and Probability Distribution

Text: WMM, Chapter 3. Sections 3.1-3.3, 3.5

Random variable

A random variable is a function that associates a real number with each element in a sample space. (Definition 3.1, p.51) WMM uses a capital letter for random variables.

Discrete Sample Space

If a sample space contains a finite number of possibilities or an unending sequences with as many elements as there are whole numbers, it is called a discrete sample space. (Definition 3.2, p.53)

Example: You flip two coins. Y is a random variable that counts the number of heads. The possible results are:

Result	y
HH	2
HT	1
TH	1
TT	0

This sample space is discrete because there are a finite number of possible outcomes.

Continuous Sample Space

If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a discrete sample space. (Definition 3.3, p.53)

Example: You drive a car with five gallons of gas. y is a random variable that represents the distance traveled. The possible results are infinite because even if the car averaged 20 miles per gallon, it could go 100.0 miles, 100.1, 100.01, 100.001, 100.0001 miles. The sample space is as infinite as real numbers.

Probability distribution function (PDF)

The function, f(x) is a probability distribution function of the discrete random variable x, if for each possible outcome a, (definition 3.4, p. 54)

$$\begin{aligned} &f(x) \geq 0 \\ &\sum_x f(x) = 1 \\ &P(x = a) = f(a) \end{aligned} \tag{4.1}$$

Example: (example 3.3, p. 54) 8 computers are shipped to a retail outlet, 3 of which are defective. If a school purchases 2 computers, find the probability distribution for the number of defective computers bought by the school.

In order to solve this problem, first define the random variable and the range of the random variable. The random variable, x, is equal to the number of defective computers bought by the school. The random variable, x, can take on values of 0, 1, and 2. Those are the only number of defective computers the school can buy, given that they are only buying two computers.

The next step is to determine the size of the sample space. The number of ways that 2 can be taken from 8 without replacement is $\binom{8}{2} = 28$. We use the formula for combinations because the order does not matter. This is the total number of combinations of computers that the school can buy.

Third, the probability of a particular outcome is equal to the number of ways to get that outcome over the total number of ways:

$$f(x) = P(x = a) = \frac{\text{ways of getting a}}{\text{total ways}}$$

$$f(0) = P(x = 0) = \frac{\binom{3}{0}\binom{5}{2}}{28} = \frac{10}{28}$$

$$f(1) = P(x = 1) = \frac{\binom{3}{1}\binom{5}{1}}{28} = \frac{15}{28}$$

$$f(2) = P(x = 2) = \frac{\binom{3}{2}\binom{5}{0}}{28} = \frac{3}{28}$$

In each of these cases, we obtained the numerator, the number of ways of getting outcome a, by using the combination rule and the generalized multiplication rule. There are $\binom{3}{a}$ ways of choosing a defective computers from 3 defective computers. There are $\binom{5}{2-a}$ ways of choosing (2-a) good computers from 5 good computers. We use the generalized multiplication rule to get the number of ways of getting both of these outcomes.

So we have the PDF, f(x), defined for all possible values of x. We have solved the problem.

(Note: If someone asked for the probability for getting 3 (or any number other than 0, 1, or 2) defective computers, then the probability is zero and f(3)= 0.

Testing a discrete PDF for legitimacy

If you are asked to determine if a given PDF is legitimate, you are required to verify the three criteria in equation (4.1). Generally, the third criterion is given in the problem statement, so you only have to check the first 2 criteria.

The first criteria, $f(x) \geq 0$, can most easily be verified by plotting f(x) and showing that it is never negative. The second criteria, $\sum_x f(x) = 1$, can most easily be verified by direct summation of all f(x).

Normalizing a discrete PDF

PDF's must satisfy $\sum_x f(x) = 1$. Sometimes, you have the functional form of the PDF and you simply need to force it to satisfy this criteria. In that case you need to normalize the PDF so that it sums to unity.

example:

Find the value of c that normalizes the PDF.

$$f(x) = c\left({}_4P_x\right) \text{ for } x = 0, 1, 2, 3, \& 4$$

To normalize:

$$\sum_x f(x) = 1 = \sum_{x=0}^{4} f(x) = f(0) + f(1) + f(2) + f(3) + f(4)$$

$$\sum_x f(x) = 1 = \sum_{x=0}^{4} c\left({}_4P_x\right) = c\sum_{x=0}^{4}\left({}_4P_x\right) = c\left({}_4P_0 + {}_4P_1 + {}_4P_2 + {}_4P_3 + {}_4P_4\right) = c(1 + 4 + 12 + 24 + 24)$$

$$1 = c(65)$$

$$c = \frac{1}{65}$$

So the normalized PDF is

$$f(x) = \frac{1}{65}\left({}_4P_x\right)$$

Discrete Cumulative distribution

The function, F(x) of a discrete random variable X with the probability distribution, f(x), is (definition 3.5, p. 55)

$$F(a) = P(x \leq a) = \sum_{x \leq a} f(x) \quad \text{for} \quad -\infty < x < \infty \tag{4.2}$$

The cumulative distribution is the probability that x is less than or equal to a.

Example: In the above example, regarding the school purchasing computers, we can obtain the cumulative distribution directly:

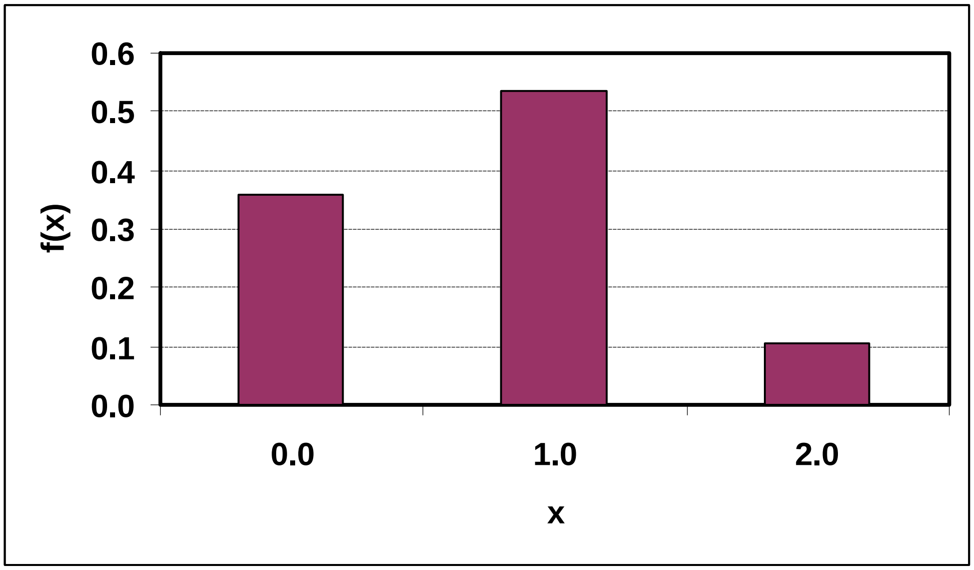
$F(0) = f(0) = 10/28$,
$F(1) = f(0)+f(1)=25/28$,
$F(2)=f(0)+f(1)+f(2)=1$

Note: The cumulative distribution is always monotonically increasing, with x.

Probability Histogram:

A probability histogram is a graphical representation of the distribution of a discrete random variable.

Example: The histogram for the example above is



The histogram gives a quick visual feeling about what is the most likely outcome and the shape of the distribution.

Probability density function (PDF)

The probability distribution functions of discrete variables are called probability density functions when applied to continuous variables. Both have the same meaning and can be abbreviated commonly as PDF's. Probability density functions satisfy the same three criteria as probability distributions, namely

$$f(x) \geq 0 \quad \text{for all } x \in R$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{4.3}$$

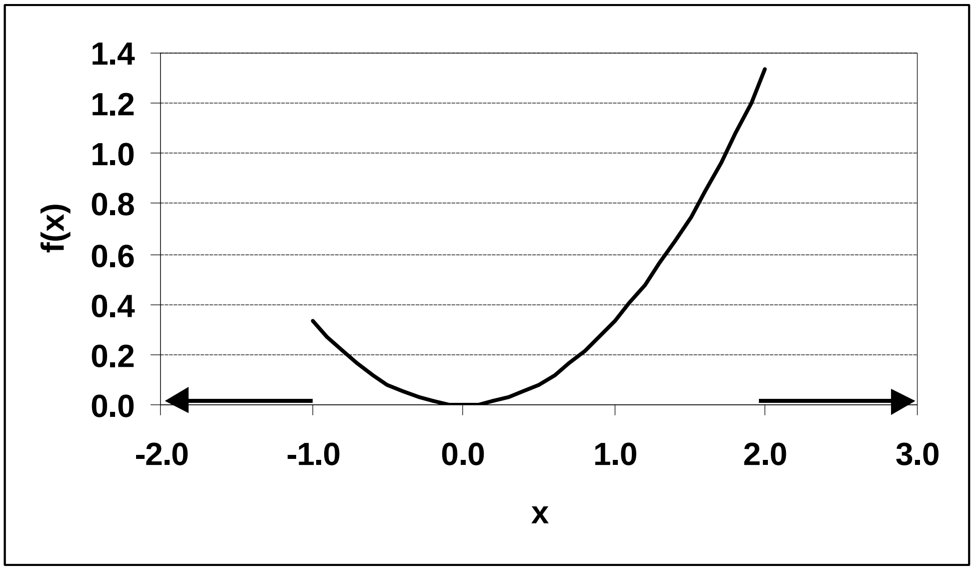
$$P(a < x < b) = \int_{a}^{b} f(x)dx$$

The probability of finding an exact point on a continuous random variable is zero.

Example: A probability density function has the form

$$f(x) = \begin{cases} \frac{x^2}{3} & \text{for } -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

A plot of the probability density distribution is shown below.



This plot is the continuous analog of the discrete histogram.
The probability of finding an x between a and b is by (equation 4.3)

$$P(a < x < b) = \int_a^b f(x)dx = \begin{cases} \int_a^b \frac{x^2}{3} dx & \text{for } -1 < x < 2 \\ \int_a^b 0 dx & \text{otherwise} \end{cases}$$

$$P(a < x < b) = \int_a^b f(x)dx = \begin{cases} \left(\frac{b^3}{9} - \frac{a^3}{9}\right) & \text{for } -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $P(-1 < x < 2)$

$$P(-1 < x < 2) = \int_{-1}^{2} f(x)dx = \left(\frac{2^3}{9} - \frac{(-1)^3}{9}\right) = 1$$

(b) Find $P(-\infty < x < \infty)$

We cannot integrate over discontinuities in a function. Therefore, we must break-up the integral over continuous parts.

$$P(-\infty < x < \infty) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 0 + \left(\frac{2^3}{9} - \frac{(-1)^3}{9}\right) + 0 = 1$$

Here we see that actually it is not practically necessary to integrate over the parts of the function where f(x)=0, because the integral over those ranges is also 0. In general practice, we just need to perform the integration over those ranges where the PDF, f(x), is non-zero.

(c) Find $P(-\infty < x < 0)$

$$P(-\infty < x < 0) = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{0} f(x)dx = 0 + \left(\frac{0^3}{9} - \frac{(-1)^3}{9}\right) = \frac{1}{9}$$

(d) Find $P(0 < x < 1)$

$$P(0 < x < 1) = \int_{0}^{1} f(x)dx = \left(\frac{1^3}{9} - \frac{0^3}{9}\right) = \frac{1}{9}$$

Testing a continuous PDF for legitimacy

If you are asked to determine if a given PDF is legitimate, you are required to verify the three criteria in equation (4.3). Generally, the third criterion is given in the problem statement, so you only have to check the first 2 criteria.

The first criteria, $f(x) \geq 0$, can most easily be verified by plotting f(x) and showing that it is never negative. The second criteria, $\int_{-\infty}^{\infty} f(x)dx = 1$, can most easily be verified by direct integration of f(x).

Normalizing a continuous PDF

Continuous PDF's must satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$. Sometimes, you have the functional form of the PDF and you simply need to force it to satisfy this criteria. In that case you need to normalize the PDF so that it sums to unity.

example:

Find the value of c that normalizes the PDF.

$$f(x) = \begin{cases} cx^2 & \text{for } -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

To normalize:

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \int_{-1}^{2} cx^2 dx = c\left.\frac{x^3}{3}\right|_{-1}^{2} = c\left(\frac{9}{3}\right) = 3c$$

$$c = \frac{1}{3}$$

So the normalized PDF is

$$f(x) = \begin{cases} \dfrac{x^2}{3} & \text{for } -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Continuous Cumulative distributions

The cumulative distribution F(x) of a continuous random variable x with density function f(x) is

$$F(a) = P(x \le a) = \int_{-\infty}^{a} f(x)dx \quad \text{for} \quad -\infty < x < \infty \tag{4.4}$$

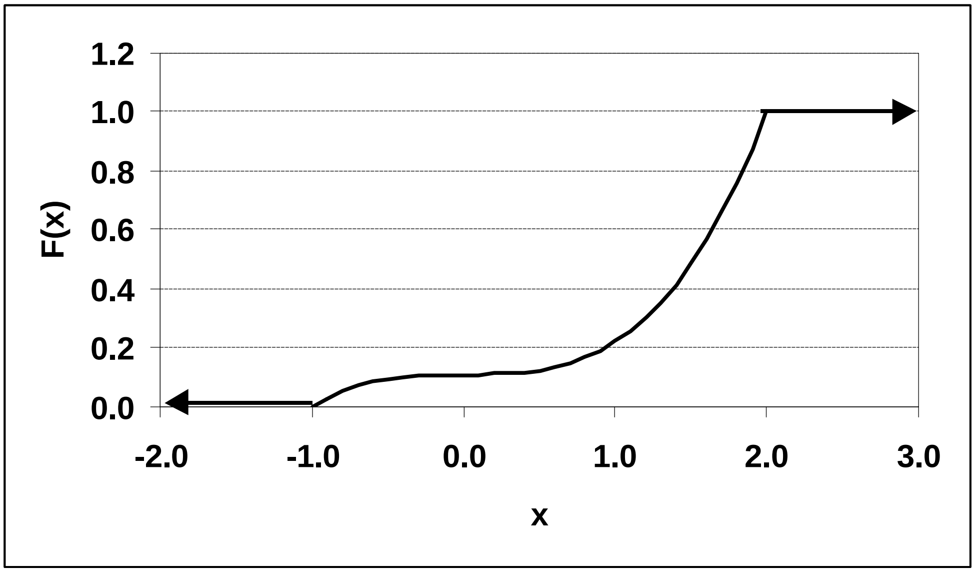
This function gives the probability that a randomly selected value of the variable x is less than a. The implicit lower limit of cumulative distribution is negative infinity.

$$F(a) = P(-\infty \le x \le a) = P(x \le a)$$

Example: Using the example PDF given above, we have that the continuous cumulative distribution is given by

$$F(a) = P(x \le a) = \int_{-\infty}^{a} f(t)dt = \begin{cases} 0 & \text{for } x < -1 \\ \left(\dfrac{a^3}{9} + \dfrac{1^3}{9}\right) & \text{for } -1 < x < 2 \\ 1 & \text{for } x > 2 \end{cases}$$

A plot of the continuous cumulative distribution is shown below:



Signs

In the above section we have defined a specific function for the probability that x is less than or equal to a, namely the cumulative distribution. But what about when x is greater than a, or strictly less than a, etc. Here, we discuss those possibilities.

Consider the fact that the probability of all outcomes must sum to one. Then we can write (regardless of whether the PDF is discrete or continuous)

$$P(x < a) + P(x = a) + P(x > a) = 1$$

Using the union rule we can write:

$$P(x \le a) = P[(x < a) \cup (x = a)] = P(x < a) + P(x = a) + P[(x < a) \cap (x = a)]$$

The intersection is zero, because x cannot equal a and be less than a, so

$$P(x \le a) = P[(x < a) \cup (x = a)] = P(x < a) + P(x = a)$$

Similarly

$$P(x \ge a) = P[(x > a) \cup (x = a)] = P(x > a) + P(x = a)$$

Using these three rules, we can create a generalized method for obtaining any arbitrary probability. On the other hand, we can use the rules to create way to obtain any probability from just the cumulative distribution function. (This will be important later when we use PDF's for which only the cumulative distribution function is given in the appendices of WMM.) Regardless of which method you use, you will obtain the same answer.

Discrete case: two ways to obtain probabilities.

Probability	Definition	from cumulative PDF
$P(x = a)$	$f(a)$	$P(x \le a) - P(x \le a-1)$
$P(x \le a)$	$\sum_{x \le a} f(x)$	$P(x \le a)$
$P(x < a)$	$\sum_{x < a} f(x)$	$P(x \le a-1)$
$P(x \ge a)$	$\sum_{x \ge a} f(x)$	$1 - P(x \le a-1)$
$P(x > a)$	$\sum_{x > a} f(x)$	$1 - P(x \le a)$

The continuous case has one important difference. In the continuous case, the probability of a random variable x equaling a single value a is zero. Why? Because the probability is a ratio of the number of ways of getting a over the total number of ways in the sample space. There is only one way to get a, namely x=a. But in the denominator, there is an infinite number of values of x, since x is continuous. Therefore, the P(x=a)=0. We can show this using the definition is we write,

$$P(x = a) = \int_a^a f(x)dx = 0 \qquad \text{for continuous PDF's only.}$$

One consequence of this is that

$$P(x \le a) = P(x < a) + P(x = a) = P(x < a)$$

$$P(x \ge a) = P(x > a) + P(x = a) = P(x > a)$$

The probability of $x \le a$ is the same as $x < a$. Likewise, the probability of $x \ge a$ is the same as $x > a$. This fact makes the continuous case easy to generate.

Continuous case: two ways to obtain probabilities.

Probability	Definition	from cumulative PDF
$P(x = a)$	$\int_a^a f(x)dx = 0$	0
$P(x \le a)$ or $P(x < a)$	$\int_{-\infty}^a f(x)dx$	$P(x \le a)$
$P(x \ge a)$ or $P(x > a)$	$\int_a^{\infty} f(x)dx$	$1 - P(x \le a)$

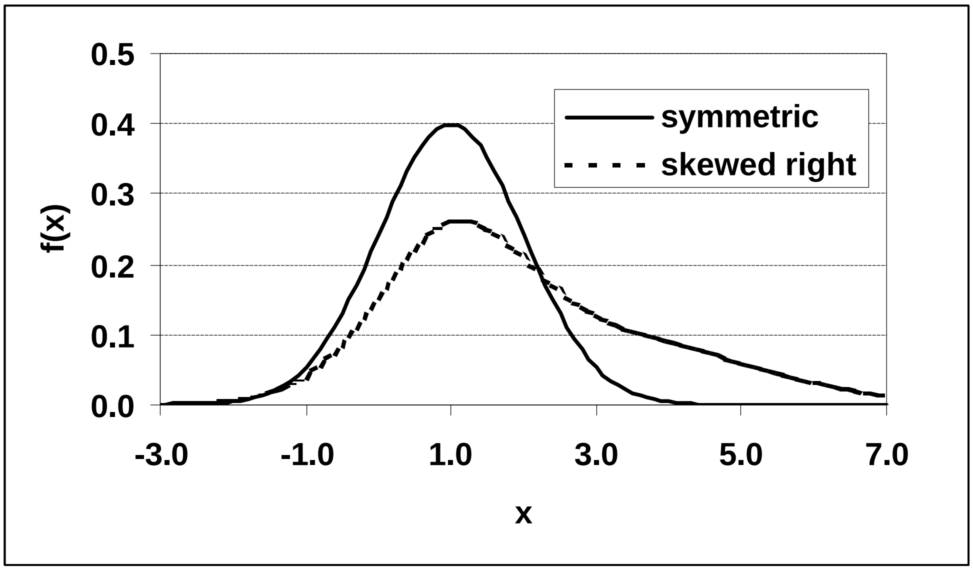
Symmetric (p. 67)

A probability density distribution is said to be symmetric if it can be folded along a vertical axis so that the two sides coincide.

Skew (p. 67)

A probability density distribution is said to be skewed if it is not symmetric.

Example:



empirical distributions

Joint Probability Distribution (Definition 3.8, p. 70)

The function f(x,y) is a joint probability distribution or probability mass function of the discrete random variable X and Y if

$$f(x,y) \geq 0$$
$$\sum_x \sum_y f(x,y) = 1 \qquad (4.5)$$
$$P(x = a \cap y = b) = f(a,b)$$

This is just the 2 variable extension of equation (4.1). Note that the Joint PDF gives the intersection of the probability.

The extension of the cumulative discrete probability distribution, equation (4.2) is that for any region A in the x-y plane,

$$F(a,b) = P(x \leq a \cap y \leq b) = \sum_{x \leq a} \sum_{y \leq b} f(x,y) \qquad (4.6)$$

That is to say, the probability that a result (x,y) is inside an arbitrary area, A, is equal to the sum of the probabilities for all of the discrete events inside A.

Example: discrete Joint PDF

f(x,y) is given by the table:

x	y	1	2	3
0		1/20	2/20	3/20
1		4/20	1/20	2/20
2		2/20	2/20	3/20

$$P(x = 1 \cap y = 2) = f(1,2) = \frac{1}{20}$$

Joint Density Function (Definition 3.9, p. 71)

The distribution of continuous variables can be extended in an exactly analogous manner as was done in the discrete case. The function f(x,y) is a Joint Density Function of the continuous random variables, x and y, if

$$f(x,y) \geq 0 \qquad \text{for all } x, y \in R$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \qquad (4.7)$$

$$P[(x,y) \in A] = \iint_A f(x,y)dxdy$$

That third equation takes a specific form, depending on the shape of the Area A. For a rectangle, it would look like:

$$P(a < x < b \cap c < y < d) = \int_c^d \int_a^b f(x,y)dxdy$$

Naturally, the cumulative distribution of the single variable case can also be extended to 2-variables.

$$F(x,y) = P(x \leq a \cap y \leq b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)dxdy \qquad (4.8)$$

Example:

Given the PDF, find $P(0 \leq x \leq 0.5 \cap 0.5 \leq y \leq 1)$

$$f(x,y) = \begin{cases} \frac{2}{5}(2x + 3y) & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0 < x < \frac{1}{2} \cap \frac{1}{2} < y < 1) = \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \frac{2}{5}(2x + 3y)dxdy$$

$$= \int_{0.5}^{1} \left[\frac{2x^2}{5} + \frac{6xy}{5} \right]_0^{0.5} dy = \int_{0.5}^{1} \left(\frac{1}{10} + \frac{3y}{5} \right) dy = \left[\frac{y}{10} + \frac{3y^2}{10} \right]_{0.5}^{1} = \frac{11}{40}$$

At this point, we should point out two things. First, we have presented eight equations, (4.1) to (4.8). However there are really only 2 equations, the requirements for the probability distribution and the definition of the cumulative probability distribution. We have shown these 2 equations for 4 cases; (i) discrete, one variable, (ii) continuous one variable, (iii) discrete, two variable, and (iv) continuous 2 variable. Re-examine these 8 equations to make sure that you see the similarities.

The second point is that we are stopping at two variables. However, discrete and continuous probability distributions can be functions of an arbitrary number of variables.

Marginal Distributions (Definition 3.10, p. 72)

Marginal distributions give us the probability of obtaining one variable outcome regardless of the value of the other variable. Marginal distributions are needed to calculate conditional probabilities. The marginal distributions of x alone and of y alone are

$$g(x) = \sum_y f(x,y) \quad \text{and} \quad h(y) = \sum_x f(x,y) \qquad (4.9)$$

for the discrete case and

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x,y)dx \qquad (4.10)$$

for the continuous case.

discrete example: The joint density function is given by the following table.

x	y	1	2	3
0		1/20	2/20	3/20
1		4/20	1/20	2/20
2		2/20	2/20	3/20

We need the marginal distribution of x at all possible values of x:

g(x=0) = f(0,1) + f(0,2) + f(0,3) = 6/20
g(x=1) = f(1,1) + f(1,2) + f(1,3) = 7/20
g(x=2) = f(2,1) + f(2,2) + f(2,3) = 7/20

We need the marginal distribution of y at all possible values of y:

h(y=1) = f(0,1) + f(1,1) + f(2,1) = 7/20
h(y=2) = f(0,2) + f(1,2) + f(2,2) = 5/20
h(y=3) = f(0,3) + f(1,3) + f(2,3) = 8/20

Continuous example: The joint density function is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find g(x) and h(y) for this joint density function.

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{0} 0dy + \int_{0}^{1} \frac{2}{5}(2x+3y)dy + \int_{1}^{\infty} 0dy$$
$$= \frac{2}{5}\left(2xy + \frac{3}{2}y^2\right)\Big|_0^1 = \frac{4x}{5} + \frac{3}{5}$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{1} \frac{2}{5}(2x+3y)dx + \int_{1}^{\infty} 0dx$$
$$= \frac{2}{5}\left(x^2 + 3yx\right)\Big|_0^1 = \frac{2}{5} + \frac{6y}{5}$$

These marginal distributions themselves satisfy all the properties of a probability density distribution, namely the requirements in equation (4.3). The physical meaning of the marginal distribution functions are that they give the individual effects of x and y separately.

Conditional Probability (Definition 3.11, p. 74)

Discrete Conditional Probability

Let x and y be two discrete random variables. The conditional distribution of the random variable y=b, given that x=a, is

$$f(y=b|x=a) = \frac{f(x=a, y=b)}{g(x=a)} \quad \text{where } g(a) > 0 \tag{4.11}$$

Similarly, the conditional distribution of the random variable x=a, given that y=b, is

$$f(x = a | y = b) = \frac{f(x = a, y = b)}{h(y = b)} \quad \text{where } h(b) > 0 \tag{4.12}$$

You should see that this conditional distribution is simply the application of the definition of the conditional probability, which we learned earlier was

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad \text{for } P(A) > 0 \tag{1.5}$$

Example: Given the discrete PDF for which you calculated the marginal distributions above, find

(a) $f(y = 2 | x = 2)$

(b) $f(x = 1 | y \le 2)$

(a) $f(y = 2 | x = 2)$. Using the conditional probability definition:

$$f(y = b | x = a) = \frac{f(x = a, y = b)}{g(x = a)}$$

We already have the denominator: g(x=2) = 7/20. The numerator is f(x=2,y=2) = 2/20. Therefore, the conditional probability is:

$$f(y = 2 | x = 2) = \frac{2/20}{7/20} = \frac{2}{7}$$

(b) $f(x = 1 | y \le 2)$

$$f(x = 1 | y \le 2) = \frac{f(x = 1, y \le 2)}{h(y \le 2)}$$

The numerator is the sum over all values of f(x,y) for which x=1, and $y \le 2$. So

$$f(x = 1, y \le 2) = f(1,1) + f(1,2) = \frac{4}{20} + \frac{1}{20} = \frac{5}{20}$$

The denominator is the sum over all h(y) for $y \le 2$

$$h(y \le 2) = h(1) + h(2) = \frac{7}{20} + \frac{5}{20} = \frac{12}{20}$$

Therefore,

$$f(x = 1 | y \le 2) = \frac{5/20}{12/20} = \frac{5}{12}$$

Continuous Conditional Probability

Let x and y be two discrete continuous variables. The conditional distribution of the random variable c<y<d, given that a<x<b, is

$$P(c < y < d | a < x < b) = \frac{\int_c^d \int_a^b f(x, y) dx dy}{\int_a^b g(x) dx} \quad \text{where} \int_a^b g(x) dx > 0 \tag{4.11}$$

Similarly, the conditional distribution of the random variable a<x<b, given that c<y<d, is

$$P(a < x < b | c < y < d) = \frac{\int_c^d \int_a^b f(x, y) dx dy}{\int_c^d h(y) dy} \quad \text{where} \int_c^d h(y) dy > 0 \tag{4.12}$$

Example: Using the joint density function and the marginal distributions from the example above, calculate: P(0<X<0.5|0.5<y<1).

$$P(0 < X < 0.5 \,|\, 0.5 < y < 1) = \frac{P(0 < X < 0.5 \cap 0.5 < y < 1)}{P(0 < X < 0.5)}$$

$$P(0 < X < 0.5 \,|\, 0.5 < y < 1) = \frac{\int_{0.5}^{1} \int_{0}^{0.5} f(x, y) dx dy}{\int_{0.5}^{1} h(y) dy}$$

We calculated the numerator in an example above (pg.11& 12), and it had a numerical value of 11/40.

The denominator is:

$$\int_{0.5}^{1} h(y) dy = \int_{0.5}^{1} \left(\frac{2}{5} + \frac{6y}{5} \right) dy = \left[\frac{2y}{5} + \frac{6y^2}{10} \right]_{0.5}^{1} = \frac{13}{20}$$

The conditional probability is then

$$P(0 < X < 0.5 \mid 0.5 < y < 1) == \frac{\frac{11}{40}}{\frac{13}{20}} = \frac{11}{26}$$

Statistical Independence (Definition 3.12, p. 77)

Let x and y be two random variables, discrete or continuous, with joint probability distribution f(x,y) and marginal distribution g(x) and h(y), respectively. The random variables x and y are said to be statistically independent iff (if and only if)

$$f(x,y) = g(x)h(y) \tag{4.13}$$

for all possible values of (x,y).
This should be compared with the rule for independence of probabilities:

$P(A \cap B) = P(A)P(B)$ iff and A and B are independent events.

Example: In the continuous example given above, determine whether x and y are statistically independent random variables.

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \frac{4x}{5} + \frac{3}{5} \quad \text{and} \quad h(y) = \frac{2}{5} + \frac{6y}{5}$$

$$g(x)h(y) = \left(\frac{4x}{5} + \frac{3}{5}\right)\left(\frac{2}{5} + \frac{6y}{5}\right) = \frac{1}{25}(8x + 24xy + 6 + 18y)$$

This is not equal to the joint probability density distribution. Therefore, the variables are not statistically independent.