Midterm Examination Number Two
Administered: Wednesday, March 3, 1999

In all relevant problems: WRITE DOWN THE FORMULA YOU USE, BEFORE YOU USE IT.
ALL PROBLEMS ARE WORTH 2 POINTS.

ALL PROBLEM PARTS ARE WORTH 2 POINTS. THE EXAM HAS 60 POINTS.

Problem 1 to 8. Each problem is worth 6 points. For each problem:
(a) Name the PDF you choose to employ. - 2 points
(b) Identify the numerical values for all parameters and variables, which are arguments in the PDF. - 2 points.
(c) Find the probability, expectation value, or statistic requested. - 2 points.

Problem 1. In a particular liquid product, opacity is a tell-tale symptom of Unacceptable purity. A simple optical sensor is used to measure the opacity of samples of the product in clear glass vials, rejecting samples based on a company policy on minimum purity standards. The optical sensor historically detects 85% of the Unacceptable samples. During the testing of 20 known Unacceptable samples, what is the probability that the optical sensor rejects 15 or fewer?

(a) Binomial PDF.

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x}$$

(b) $p = 0.85$ and $q = 1 - p = 0.15$ and $n = 20$ and $x \le 15$

(c) $P(X \le 15) = B(15;20,0.85) = 0.02068$ From Table A.1
interpolating between the values for p=0.8 and 0.9

Problem 2. For the same optical sensor as in Problem 1, we test samples as they move by on a conveyor belt. Historically our plant rejects 10 samples per hour. What is the probability that the optical sensor rejects more than 5 samples in half an hour?

(a) Poisson PDF.

$$p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad \text{for } x = 0,1,2...$$

(b) $\lambda = 10$ and $t = 0.5$ and $x > 5$

(c) $P(x > 5) = 1 - P(x \le 5) = 1 - p(5;5) = 1 - 0.6160 = 0.3840$ From Table A.2

Problem 3. For the same optical sensor as in Problem 1, we test a series of known defective samples. What is the probability that the optical sensor makes its seventh rejection on the tenth sample?

(a) negative binomial PDF.

$$b^*(x;k,p) = \binom{x-1}{k-1} p^k q^{x\text{-}k} \quad \text{for } x = k, k+1, k+2...$$

(b) $p = 0.85$ and $q = 1 - p = 0.15$ and $x = 10$ and $k = 7$

(c) $P(X = 10) = \binom{10-1}{7-1}(0.85)^7(0.15)^{10-7} = 0.0909$

Problem 4. In a collection of 20 samples of our liquid product, we know we have 3 Unacceptable samples and 17 Acceptable samples. If we were to simply randomly select 3 samples, what would be the probability that we got at least 2 Unacceptable samples?

(a) hypergeometric PDF.

$$h(x;N,n,k) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad \text{for } x = 0,1,2...n$$

(b) $k = 3$, $N = 20$, $n = 3$, $x \geq 3$

(c) $P(x \geq 2) = P(x = 2) + P(x = 3)$

$$P(x = 2) = \frac{\binom{3}{2}\binom{20-3}{3-2}}{\binom{20}{3}} = \frac{3 \cdot 17}{1140} = 0.0447$$

$$P(x = 3) = \frac{\binom{3}{3}\binom{20-3}{3-3}}{\binom{20}{3}} = \frac{1 \cdot 1}{1140} = 0.0009$$

$$P(x \geq 2) = P(x = 2) + P(x = 3) = 0.0456$$

Problem 5. In a recent advancement to optical sensors which detect opacity of a liquid product, sensors are now able to classify product into three grades: grade A, grade B, and Unacceptable. Historically, we know that 80% of our product is grade A, 15% is grade B, and 5% is Unacceptable. During the testing of 10 unknown samples, what is the probability that 7 samples are grade A, 2 samples are grade B, and 1 sample is Unacceptable?

(a) multinomial PDF.

$$m(\{x\};n,\{p\},k) = \binom{n}{x_1, x_2 ... x_k} \prod_{i=1}^{k} p_i^{x_i}$$

(b) $p_1 = 0.8$, $p_2 = 0.15$, $p_3 = 0.05$, $k = 3$

$x_1 = 7$, $x_2 = 2$, $x_3 = 1$, $n = 10$

(c) $P(\{X\} = \{7,2,1\}) = \binom{10}{7,2,1} \prod_{i=1}^{3} p_i^{x_i} = 0.0849$

Problem 6. Our grade A liquid product, on average, contains an impurity concentration of 100 ppm, with a standard deviation of 50 ppm. What is the probability that a give sample of the product has an impurity concentration greater than 130 ppm?

(a) Normal PDF.

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

(b) $\mu = 100$ and $\sigma = 50$ and $x > 130$

(c) $z = \frac{x-\mu}{\sigma} = \frac{130-100}{50} = 0.6$

$$P(X > 130) = P(Z > 0.6) = 1 - P(Z < 0.6)$$ From Table A.3

$$= 1 - 0.7257 = 0.2743$$

Problem 7. With our advanced optical sensors, we again collect 20 samples, 16 of which are grade A, 3 are grade B, and 1 is Unacceptable. If we were to randomly select 3 samples, what would be the probability that we got 1 Unacceptable sample, 1 grade B sample, and 1 grade A sample?

(a) multivariate hypergeometric PDF.

$$h(\{x\};N,n,\{a\}) = \frac{\binom{a_1}{x_1}\binom{a_2}{x_2}\binom{a_3}{x_3}\cdots\binom{a_k}{x_k}}{\binom{N}{n}}$$

(b) $a_1 = 16$, $a_2 = 3$, $a_3 = 1$, $N = \sum a_i = 20$

$x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $n = \sum x_i = 3$

(c) $$P(\{X\} = \{0,1,0,1\}) = \frac{\binom{16}{1}\binom{3}{1}\binom{1}{1}}{\binom{20}{3}} = \frac{48}{1140} = 0.0421$$

Problem 8. Our liquid product is a fungicide for tomato plants. The fungicide has a mean effective life-time of 21 days. We apply the fungicide to 16 tomato plants. What is the probability that at least 8 of our tomato plants are protected by the fungicide after 28 days?

(a) Binomial PDF and exponential PDF.

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x} \text{ and } f_e(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b) For exponential: $\beta = 21$, $x \ge 28$

For binomial: $p = ?$ and $q = 1 - p$ and $n = 16$ and $x \ge 8$

(c) $p = P(t_i < t) = \int_{t_i}^{\infty} f_e(t;\beta)dt = \int_{t_i}^{\infty} \frac{1}{\beta} e^{-t/\beta}dt = e^{-t_i/\beta} = e^{-\frac{28}{21}} = 0.2636$

$P(x \geq 8) = 1 - P(x \leq 7) = 1 - B(7;16,0.2636) = 1 - 0.9729 = 0.0271$

using table A.1 and approximating p = 0.25

Problem 9.

Our salesmen tell a tomato-farmer that our fungicide has lasts on average 21 days before it needs to be reapplied, with a standard deviation of 4 days, owing to rain. Our competitor's salesmen tell the same tomato farmer that there fungicide lasts on average 28 days with a standard deviation of 2 days. We don't believe our competitors claim and to test it, we take 32 tomato plants and apply our fungicide to half of them and the competitor's fungicide to the other half. Our analysis of the two samples indicates that the mean life of our fungicide is 22 days with a sample standard deviation of 3 days, and that the mean life of our competitor's fungicide is 25 days with a sample standard deviation of 6 days.

(a) Find a 95% confidence interval for the difference in the two companies' product's average life-times.

(b) Does the claimed average life-time difference fall within this confidence interval?

(c) Find a 98% confidence interval for the ratio of our product variance to our competitor's product variance.

(d) Does the claimed population variance ratio fall within this confidence interval?

(e) What PDFs did you use for parts (a) and (c)?

(a) To estimate the difference of means with variance known, use the normal distribution.

$\alpha = 0.025$, $-z_\alpha = -z_{0.025} = -1.96$ From table A.3

$$P\left[\left(\overline{X}_1 - \overline{X}_2\right) - z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < (\mu_1 - \mu_2) < \left(\overline{X}_1 - \overline{X}_2\right) + z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] = 1 - 2\alpha$$

$$P\left[3 - 1.96\sqrt{\frac{2^2}{16} + \frac{4^2}{16}} < (\mu_1 - \mu_2) < 3 + 1.96\sqrt{\frac{2^2}{16} + \frac{4^2}{16}}\right] = 0.95$$

$$P[0.8 < (\mu_1 - \mu_2) < 5.2] = 0.95$$

(b) No, the sampling data $\mu_1 - \mu_2 = 28 - 21 = 7$ does not fall within this confidence interval.

(c) Use the F-distribution to describe the distribution of the ratio of two variances.

$v_1 = 15$, $v_2 = 15$, $\alpha = 0.01$, $f_\alpha(v_1, v_2) = 3.52$ from table A.6

$f_\alpha(v_2, v_1) = 3.22$

$$P\left[\frac{S_1^2}{S_2^2}\frac{1}{f_\alpha(v_1, v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} f_\alpha(v_2, v_1)\right] = 1 - 2\alpha$$

$$P\left[\frac{6_1^2}{3_2^2}\frac{1}{3.52} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{6_1^2}{3^2} 3.52\right] = 0.98$$

$$P\left[1.14 < \frac{\sigma_1^2}{\sigma_2^2} < 14.08\right] = 0.98$$

(d) No, the ratio of variances that we were given, $\frac{\sigma_1^2}{\sigma_2^2} = \frac{2^2}{4^2} = 0.25$, does not fall into this confidence interval.

(e) We used the normal and F-distributions for part (a) and (c) respectively.

You got two more points for showing up to make the point total 60.