Exam I
Administered: Monday, September 15, 2003
20 points

For each problem part: 0 points if not attempted or no work shown,
1 point for partial credit, if work is shown,
2 points for correct numerical value of solution

**Problem 1. (10 points)**

In a statistical study of automobiles, it is found that SUVs are involved in 55% of automobile accidents involving roll-overs. When an SUV was involved, 18% of the roll-overs resulted in a fatality. When an automobile other than an SUV was involved, 24% of the roll-overs resulted in a fatality. Answer the following questions. Where appropriate, report to 4 significant figures.

(a) Draw a Venn Diagram of the sample space for the type of automobile and fatal outcome of the accident.
(b) What is the probability that an accident involved an SUV and resulted in a fatality?
(c) What is the probability that an accident resulted in a fatality?
(d) What is the probability that an accident occurred in a vehicle other than an SUV given that it resulted in a fatality?
(e) What is the probability that an accident occurred in a vehicle other than an SUV and did not result in a fatality?

**Solution:**

S = SUV
O = Other Vehicle
F = Fatality
N = No Fatality

We are given:

$$P(S) = 0.55$$
$$P(F \mid S) = 0.18$$
$$P(F \mid O) = 0.24$$

(a) Draw a Venn Diagram of the sample space for the type of automobile and fatal outcome of the accident.

| | |
|---|---|
| $S \cap F$ | $O \cap F$ |
| $S \cap N$ | $O \cap N$ |

(b) What is the probability that an accident involved an SUV and resulted in a fatality?

$$P(F \mid S) = \frac{P(F \cap S)}{P(S)}$$

$$P(F \cap S) = P(F \mid S)P(S) = 0.18 \cdot 0.55 = 0.0990$$

(c) What is the probability that an accident resulted in a fatality?

$$P(O) = 1 - P(S)$$

$$P(F \mid O) = \frac{P(F \cap O)}{P(O)}$$

$$P(F \cap O) = P(F \mid O)P(O) = P(F \mid O)[1 - P(S)] = 0.24 \cdot [1 - 0.55] = 0.1080$$

$$\begin{aligned} P(F) = P[(F \cap S) \cup (F \cap O)] &= P(F \cap S) + P(F \cap O) - P[(F \cap S) \cap (F \cap O)] \\ &= 0.0990 \quad + 0.1080 \quad - 0 \qquad\qquad = 0.207 \end{aligned}$$

(d) What is the probability that an accident occurred in a vehicle other than an SUV given that it resulted in a fatality?

$$P(O \mid F) = \frac{P(F \cap O)}{P(F)} = \frac{0.1080}{0.2070} = 0.5217$$

(e) What is the probability that an accident occurred in a vehicle other than an SUV and did not result in a fatality?

$$P(O) = P[(N \cap O) \cup (F \cap O)] = P(N \cap O) + P(F \cap O) - P[(N \cap O) \cap (F \cap O)]$$

$$P[(N \cap O) \cap (F \cap O)] = 0$$

$$P(N \cap O) = P(O) - P(F \cap O) = 0.45 - 0.1080 = 0.3420$$

**Problem 2. (10 points)**

A company manufactures rechargeable metal hydride batteries. They have performed an internal study and determined that an individual battery's lifetime, before it must be recharged, is given by the probability distribution function:

$$f(t) = \begin{cases} c \cdot \exp(-\frac{t}{60}) & \text{for } 0 \le t \\ 0 & \text{otherwise} \end{cases}$$

where t is the operating time in hours.

(a) What is the random variable in this problem, both in terms of physical interpretation and the variable used?
(b) What value of $c$ will make this PDF a legitimate function?
(c) What fraction of the batteries die within 24 hours?
(d) What fraction of the batteries die after 24 hours?
(e) What is the mean battery life?

You may find the following indefinite integrals useful. Or you may not.

$$\int a \exp(-br)\, dr = -\frac{a}{b} \exp(-br)$$

$$\int ar \exp(-br)\, dr = -\frac{a}{b^2}(br + 1)\exp(-br)$$

$$\int ar^2 \exp(-br)\,dr = -\frac{a}{b^3}\left(b^2r^2 + 2br + 2\right)\exp(-br)$$

$$\int ar^3 \exp(-br)\,dr = -\frac{a}{b^4}\left(b^3r^3 + 3b^2r^2 + 6br + 6\right)\exp(-br)$$

Also, you may find the following limit useful.

$$\lim_{x\to\infty} x \cdot \exp(-x) = 0$$

**Solution:**

(a) What is the random variable in this problem, both in terms of physical interpretation and the variable used?

The random variable, $t$, is the operating time of the battery, before it must be recharged.

(b) What value of $c$ will make this PDF a legitimate function?

$$P(-\infty < t < \infty) = 1 = \int_{-\infty}^{\infty} f(t)dt = \int_{0}^{\infty} c \cdot \exp(-\frac{t}{60})dt = \left[-60 \cdot c \cdot \exp(-\frac{t}{60})\right]_0^{\infty}$$
$$= 0 - \left[-60 \cdot c\right] = 60 \cdot c$$

$$c = \frac{1}{60} \approx 0.0166666667$$

(c) What fraction of the batteries die within 24 hours?

$$P(0 < t < 24) = \int_{0}^{24} f(t)dt = \int_{0}^{24} \frac{1}{60}\exp(-\frac{t}{60})dt = \left[-\exp(-\frac{t}{60})\right]_0^{24}$$
$$= \left[-\exp(-\frac{24}{60})\right] - \left[-\exp(-\frac{0}{60})\right] = 1 - \exp(-\frac{24}{60}) \approx 0.3297.l$$

(d) What fraction of the batteries die after 24 hours?

$$P(t > 24) = 1 - P(0 < t < 24) \approx 1 - 0.3297 = 0.6703$$

(e) What is the mean battery life?

$$\mu_t = \int_{-\infty}^{\infty} tf(t)dt = \int_{0}^{\infty} t\frac{1}{60} \cdot \exp(-\frac{t}{60})dt$$

This integral is of the form:

$$\int ar\exp(-br)\,dr = -\frac{a}{b^2}(br+1)\exp(-br)$$

where $a = \frac{1}{60}$ and $b = \frac{1}{60}$ so that

$$\mu_t = \int_{-\infty}^{\infty} tf(t)dt = \int_0^{\infty} t\frac{1}{60}\cdot\exp(-\frac{t}{60})dt = \left[-60\left(\frac{1}{60}t+1\right)\exp(-\frac{t}{60})\right]_0^{\infty}$$

$$= 0 - \left[-60\left(\frac{1}{60}0+1\right)\exp(-\frac{0}{60})\right] = 60$$

The mean life time is 60 hours.