Problem 1. (16 points)

A railroad tanker containing concentrated sulfuric acid derails near a populated area. The concentration of sulfuric acid in the air as a function of the radial position from the point of the derailment at its worst time is given by the following function, where \( r \) is in miles, and \( c \) is in ppm

\[
c(r) = \begin{cases} 
20 \exp(-2r) & \text{for } 0 \leq r \leq 10 \\
0 & \text{otherwise}
\end{cases}
\]

The probability distribution of the concentration of sulfuric acid is proportional to the concentration.

\[
f(r) = 2\pi c_0 c(r)r
\]

(a) What is the random variable in this problem, both in terms of physical interpretation and the variable used?
(b) What value of \( c_0 \) will make this PDF a legitimate function?
(c) What fraction of the sulfuric acid is located within 1 mile of the derailment?
(d) What fraction of the sulfuric acid is located beyond 1 mile of the derailment?
(e) What is the mean sulfuric acid concentration in the ten mile radius?

You may find the following indefinite integrals useful:

\[
\int ar \exp(-br) \, dr = -\frac{a}{b^2} (br + 1) \exp(-br)
\]

\[
\int ar^2 \exp(-br) \, dr = -\frac{a}{b^3} \left( b^2 r^2 + 2br + 2 \right) \exp(-br)
\]

\[
\int ar^3 \exp(-br) \, dr = -\frac{a}{b^4} \left( b^3 r^3 + 3b^2 r^2 + 6br + 6 \right) \exp(-br)
\]

Solution:

(a) What is the random variable in this problem, both in terms of physical interpretation and the variable used?

The random variable, \( r \), is the radial position. \( c(r) \) is a function of the random variable \( r \).

(b) What value of \( c_0 \) will make this PDF a legitimate function?
\[
\begin{align*}
P(-\infty < x < \infty) &= 1 = \int_{-\infty}^{\infty} f(r) dr = \int_{0}^{10} 2 \pi c_0 c(r) r dr = \int_{0}^{10} 2 \pi c_0 20 \exp(-2r) r dr
\end{align*}
\]

\[
\begin{align*}
c_0 &= \frac{1}{2 \pi 20} \int_{0}^{10} \exp(-2r) r dr = \frac{1}{2 \pi 20} \left[-\frac{1}{2^2} (2r + 1) \exp(-2r)\right]_0^{10}
\end{align*}
\]

\[
\begin{align*}
c_0 &= \frac{1}{2 \pi 20} \left[-\frac{1}{2^2} (2 \cdot 10 + 1) \exp(-2 \cdot 10)\right] - \left[-\frac{1}{2^2}\right] \\
&\approx 0.0318
\end{align*}
\]

(c) What fraction of the sulfuric acid is located within 1 mile of the derailment?

\[
\begin{align*}
P(0 < x < 1) &= \int_{0}^{1} f(r) dr = \int_{0}^{1} 2 \pi c_0 20 \exp(-2r) r dr = 2 \pi c_0 \left[-\frac{1}{2^2} (2r + 1) \exp(-2r)\right]_0^{1}
\end{align*}
\]

\[
\begin{align*}
P(0 < x < 1) &= 2 \pi c_0 \left[-\frac{1}{2^2} (2 \cdot 1 + 1) \exp(-2 \cdot 1)\right] - \left[-\frac{1}{2^2}\right] \\
&\approx 0.5940
\end{align*}
\]

(d) What fraction of the sulfuric acid is located beyond 1 mile of the derailment?

\[
\begin{align*}
P(x > 1) &= 1 - P(0 < x < 1) \approx 1 - 0.5940 = 0.4060
\end{align*}
\]

(e) What is the mean sulfuric acid concentration in the ten mile radius?

\[
\begin{align*}
\mu_c &= \int_{-\infty}^{\infty} c(r) f(r) dr = \int_{0}^{10} c(r) 2 \pi c_0 c(r) r dr = \int_{0}^{10} (20 \exp(-2r)) 2 \pi c_0 20 \exp(-2r) r dr
\end{align*}
\]

\[
\begin{align*}
\mu_c &= \pi c_0 800 \int_{0}^{10} \exp(-4r) r dr = \pi c_0 800 \int_{0}^{10} \exp(-4r) r dr
\end{align*}
\]

\[
\begin{align*}
\mu_c &= \pi c_0 800 \left[-\frac{a}{b^2} (br + 1) \exp(-br)\right]_0^{10} = \pi c_0 800 \left[-\frac{1}{16^2} (4r + 1) \exp(-4r)\right]_0^{10}
\end{align*}
\]

\[
\begin{align*}
\mu_c &= \pi c_0 800 \left[-\frac{1}{4^2} (4r + 1) \exp(-4r)\right] - \left[-\frac{1}{4^2}\right]_0^{10} \\
&\approx 5.000 \text{ ppm}
\end{align*}
\]
Problem 2. (10 points)

Seudenol, C₇H₁₂O, is an aggregation pheromone from the female Douglas fir beetle, *Dendroctonus pseudotsugae*. The natural pheromone is a racemic mixture which is much more biologically active than either single enantiomer. The two enatiomers, (R)-seudenol and (S)-seudenol, are shown in the figure below.

We are studying two alternative methods, method A and method B to synthesize this mixture. Method A was used to generate 40% of the product. Method A produced 64% (R)-seudenol. Method B produces 76% (R)-seudenol. Answer the following questions. Where appropriate, report to 4 significant figures.

(a) Draw a Venn Diagram of the sample space for the process and classification of the molecules in the product.

(b) What is the probability that a molecule was synthesize using method A and is (R)-seudenol?

(c) What is the probability that a molecule is (R)-seudenol?

(d) What is the probability that a molecule was generated using method B given that it is (R)-seudenol?

(e) What is the probability that a molecule was synthesize using method B and is (S)-seudenol?

Solution:

We are given:

\[
\begin{align*}
P(A) &= 0.40 \\
P(R \mid A) &= 0.64 \\
P(R \mid B) &= 0.76
\end{align*}
\]

(a) Draw a Venn Diagram of the sample space for the process and classification of the product.

(b) What is the probability that a molecule was synthesize using method A and is (R)-seudenol?

\[
P(R \mid A) = \frac{P(R \cap A)}{P(A)}
\]

\[
P(R \cap A) = P(R \mid A)P(A) = 0.64 \times 0.40 = 0.256
\]

(c) What is the probability that a molecule is (R)-seudenol?
P(B) = 1 − P(A)

P(R | B) = \frac{P(R \cap B)}{P(B)}

P(R \cap B) = P(R | B)P(B) = P(R | B)[1 − P(A)] = 0.76 \cdot [1 − 0.40] = 0.456

P(R) = P[(R \cap A) \cup (R \cap B)] = P(R \cap A) + P(R \cap B) − P[(R \cap A) \cap (R \cap B)]

= 0.256 + 0.456 − 0 = 0.712

(d) What is the probability that a molecule was generated using method B given that it is (R)-seudenol?

P(B | R) = \frac{P(R \cap B)}{P(R)} = \frac{0.456}{0.712} = 0.6404

(e) What is the probability that a molecule was synthesize using method B and is (S)-seudenol?

P(B) = P[(S \cap B) \cup (R \cap B)] = P(S \cap B) + P(R \cap B) − P[(S \cap B) \cap (R \cap B)]

P[(S \cap B) \cap (R \cap B)] = 0

P(S \cap B) = P(B) − P(R \cap B) = 0.6 − 0.456 = 0.144