

Exam IV: Administered: Monday December 10, 2001

Problem (1) (10 points)

Perform one complete Newton-Raphson iteration on the system of equations:

$$x + y = 4 \quad xy = 2$$

Use $(x,y) = (1/2, 7/2)$ as your initial guess.Along the way, present the Jacobian, Residual, determinant, inverse, and new estimate of $[x,y]$.**solution:**

$$f_1(x,y) = x + y - 4 = 0$$

$$f_2(x,y) = xy - 2 = 0$$

$$\underline{J} = \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} x + y - 4 \\ xy - 2 \end{bmatrix}$$

$$\underline{J}(x = \frac{1}{2}, y = \frac{7}{2}) = \begin{bmatrix} 1 & 1 \\ \frac{7}{2} & \frac{1}{2} \end{bmatrix} \quad \underline{R}(x = \frac{1}{2}, y = \frac{7}{2}) = \begin{bmatrix} x + y - 4 \\ xy - 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{7}{2} - 4 \\ \frac{1 \cdot 7}{2 \cdot 2} - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix}$$

$$\det(\underline{J}) = j_{11}j_{22} - j_{21}j_{12} = (1)(\frac{1}{2}) - (\frac{7}{2})(1) = -3$$

$$\underline{J}^{-1} = \frac{1}{\det(\underline{J})} \begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{7}{2} & 1 \end{bmatrix}$$

$$\underline{\delta x} = -\underline{J}^{-1}\underline{R} = -\frac{1}{-3} \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{7}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} \\ -\frac{1}{12} \end{bmatrix}$$

$$\underline{x}^{(1)} = \underline{x}^{(0)} + \underline{\delta x} = \begin{bmatrix} \frac{1}{2} \\ \frac{7}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{12} \\ -\frac{1}{12} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{12} \\ \frac{12}{41} \\ \frac{12}{12} \end{bmatrix}$$

Problem (2). (14 points)

Consider the linear ordinary differential equation initial value problem

$$\frac{dy}{dx} = 3x^2 \quad (2.1)$$

where $y(x_0 = 1) = y_0 = 1$

The analytical solution to this ODE IVP can be obtained by separating variables:

$$dy = 3x^2 dx \quad (2.2)$$

$$\int_{y_0}^y dy = \int_{x_0}^x 3x^2 dx \quad (2.3)$$

$$y(x) - y_0 = \int_{x_0}^x 3x^2 dx \quad (2.4)$$

$$y(x) = y_0 + \int_{x_0}^x 3x^2 dx \quad (2.5)$$

$$y(x) = y_0 + x^3 - x_0^3 \quad (2.6)$$

- Solve the ODE as it is given in equation (2.1) using the Euler Method. Use a step size of $\Delta x=1$ and report the solution at $x=3$.
- Solve the ODE as it is given in equation (2.5) using the Trapezoidal Rule. Again, use a step size of $\Delta x=1$ and report the solution at $x=3$.
- Report the analytical solution of the ODE at $x=3$.
- Which method is more accurate in this problem, the Euler Method or the Trapezoidal method?
- Why?

Solution:

- Solve the ODE as it is given in equation (2.1) using the Euler Method. Use a step size of $\Delta x=1$ and report the solution at $x=3$.

The Euler method is given by the equation

$$y_{i+1} = y_i + \Delta x \left(\frac{dy}{dx} \right)_{x=x_i}$$

Applying this formula twice and equation (2.1) and the initial condition yields

i	x	y(x)	dy/dx
0	1	1	3
1	2	4	12
2	3	16	

Therefore, $y(x=3) = 16$.

(b) Solve the ODE as it is given in equation (2.5) using the Trapezoidal Rule. Again, use a step size of $\Delta x=1$ and report the solution at $x=3$.

The Trapezoidal method is given by the equation

$$\int_a^b f(x)dx \approx I_{\text{TRAP}} = \frac{\Delta x}{2} \left[f(a) + f(b) + 2 \sum_{j=2}^n f(x_j) \right]$$

In this problem $a = 1$, $b = 3$, $f(x) = 3x^2$, $\Delta x = 1$, so $n=2$ intervals.

$$I_{\text{TRAP}} = \frac{1}{2} \left[3(1)^2 + 3(3)^2 + 2 \cdot 3(2)^2 \right] = 27$$

Using equation (2.5) and the initial condition, we see that $y(x=3) = 28$.

(c) Report the analytical solution of the ODE at $x=3$.

The analytical solution is given by substituting the initial condition into equation (2.6)

$$y(x=3) = y_0 + x^3 - x_0^3 = 1 + 3^3 - 1^3 = 27$$

(d) Which method is more accurate in this problem, the Euler Method or the Trapezoidal method?

The Trapezoidal method is more accurate because the absolute error of the trapezoidal method is 1 and the absolute error of the Euler method is 11.

(e) Why?

The trapezoidal rule is more accurate because it uses the function evaluated at $x=1$, 2, and 3. The Euler method only uses the function evaluated at $x=1$ and 2. Because the trapezoidal rule requires more information, it is able to give a more accurate approximation of the solution.

Problem (3) (8 points)

In solving the solution to $\underline{\underline{A}}\underline{\underline{x}} = \underline{\underline{b}}$, where $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & 4 \end{bmatrix}$ and $\underline{\underline{b}} = \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}$, we find the following

information on the reduced row echelon form of the $A|b$ augmented matrix.

$$\text{rref}(\underline{\underline{A}} | \underline{\underline{b}}) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

- (a) What is the determinant of A?
- (b) Does the inverse of A exist?
- (c) How many solutions exist to $\underline{\underline{A}}\underline{\underline{x}} = \underline{\underline{b}}$?

- (d) If infinite solutions exist, find the solution $\underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$.

Solution:

- (a) What is the determinant of A?

The determinant of A is zero, because the $\text{rank}(A) = 2$, which is less than the size of A (which is $n \times n$ where $n = 3$).

- (b) Does the inverse of A exist?
No. The determinant of A is zero.

- (c) How many solutions exist to $\underline{\underline{A}}\underline{\underline{x}} = \underline{\underline{b}}$?
Since the $\text{rank}(A|b) > \text{rank}(A)$, there are no solutions.

- (d) If infinite solutions exist, find the solution $\underline{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$.

There are no solutions.

Problem (4) (8 points)

You are sampling from a reactor that is creating polymer with an average molecular weight of 7.5×10^6 amu and a standard deviation of 3.0×10^6 amu.

- (a) What fraction of the product has a molecular weight less than 4.5×10^6 amu ?
- (b) What PDF did you use to solve part (a)?
- (c) 90% of the product has a molecular weight greater than what value?
- (d) What PDF did you use to solve part (c)?

Solution:

- (a) What fraction of the product has a molecular weight less than 4.5×10^6 amu ?

$$z = \frac{x - \mu}{\sigma} = \frac{4.5 \cdot 10^6 - 7.5 \cdot 10^6}{3.0 \cdot 10^6} = -1$$

$$p(x < 4.5 \cdot 10^6) = p(z < -1) = 0.1587$$

The standard normal distribution value was obtained from appendix A.3 of WMM.

- (b) What PDF did you use to solve part (a)?

I would use the normal distribution because the molecular weight can be considered to be a continuous variable. Moreover, we only know the mean and the variance, so we have enough information to use the normal distribution.

- (c) 90% of the product has a molecular weight greater than what value?

$$p(z > z_{10}) = 0.90$$

$$p(z < z_{10}) = 1 - 0.90 = 0.10$$

From table A.3 in WMM, we find that $z_{10} = -1.28$

$$z = \frac{x - \mu}{\sigma} = -1.28$$

$$x = \mu + \sigma z = 7.5 \cdot 10^6 + 3.0 \cdot 10^6(-1.28) = 3.66 \cdot 10^6$$

- (d) What PDF did you use to solve part (c)?

Same answer as in (b). Normal PDF.

Problem 5. (12 points)

A manufacturer of automobile batteries claims that their batteries last on average 8 years. You buy two batteries. The first fails in 6 years and the second fails in 3 years.

- What is the sample mean of the battery life?
- What is the sample standard deviation?
- What is the lower limit on a 95% confidence interval for the mean battery life?
- What is the upper limit on a 95% confidence interval for the mean battery life?
- Is the manufacturer's claim reasonable?
- Explain your answer to part (e).

Solution:

- What is the sample mean of the battery life?

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i = \frac{1}{2}(6 + 3) = 4.5$$

- What is the sample standard deviation?

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{1} [(6 - 4.5)^2 + (3 - 4.5)^2] = 4.5$$

$$s = \sqrt{s^2} = \sqrt{4.5} = 2.1213$$

- What is the lower limit on a 95% confidence interval for the mean battery life?
- What is the upper limit on a 95% confidence interval for the mean battery life?

To estimate the mean, variance unknown, use the t-distribution.

$v = n - 1 = 1$. First, find $t_{\alpha/2}$ for $\alpha = 0.05$ from table A.4, $t_{0.025} = 12.706$

$$P\left(\bar{X} - t_{\alpha/2} \frac{s}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2} \frac{s}{\sqrt{n}}\right) = 1 - \alpha$$

$$P\left(4.5 - (12.706) \frac{2.1213}{\sqrt{2}} < \mu < 4.5 + (12.706) \frac{2.1213}{\sqrt{2}}\right) = 0.95$$

$$P(-14.56 < \mu < 23.56) = 0.95$$

- Is the manufacturer's claim reasonable?
- Explain your answer to part (e).

The claim of 8 years is reasonable because it falls within the 95% confidence interval.

Problem (6) (4 points)

In Computer Project 2, you solved the steady state behavior of the reactor in the adiabatic and nonadiabatic mode.

- (a) Which mode yielded a higher steady state operating temperature in the reactor?
- (b) Why?

Solution:

- (a) Which mode yielded a higher steady state operating temperature in the reactor?
- (b) Why?

The adiabatic temperature is higher than the nonadiabatic temperature, because in the nonadiabatic case, the cooling jacket is removing some heat from the reactor. In the adiabatic case, none of the heat generated by the exothermic reaction is removed by the cooling jacket.