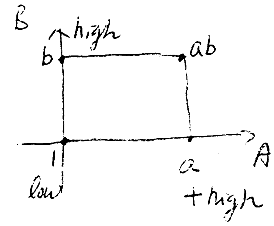
# $2^2$ Factorial Experiment

Factors : A & B

levels : High & low




| Treatment | A | B |
|---|---|---|
| (1) | − | − |
| a | + | − |
| b | − | + |
| ab | + | + |

| Treatment Combination | I | A | B | AB | Replicates | | | | Total | Av |
|---|---|---|---|---|---|---|---|---|---|---|
| | Factorial Effect | | | (Sgn(A) x sgn(B)) | | | | | | |
| (1) | + | − | − | + | 1.40 | 1.42 | 13.9 | 12.4 | | |
| a | + | + | − | − | # | # | # | # | | |
| b | + | − | + | − | # | # | # | # | | |
| ab | + | + | + | + | # | # | # | # | | |
| | | | | | | | | | $y_{..}$ | $\bar{y}_{..}$ |

= indicate the Sum of all n observations Taken at these design points.

d.o.f.

1 $$SS_A = \frac{[a + ab - (1) - b]^2}{4n}$$

1 $$SS_B = \frac{[b + ab - a - (1)]^2}{4n}$$

1 $$SS_{AB} = \frac{[ab + (1) - a - b]^2}{4n}$$

$(4n-1)$ $$SS_T = \sum_i^4 \sum_j^n Y_{ij}^2 - \frac{Y_{..}^2}{4n}$$

$4(n-1)$ $$SSE = SST - SSA - SS_B - SS_{AB}$$

Example

| Sgn (Factor Effect)? | Effect | SS | d.o.f. | MS | $F_0$ | P-Value |
|---|---|---|---|---|---|---|
| $(a+ab-(1)-b)/2n$ | A | 2.7956 | 1 | 2.7956 | 134.40 | 7 E-f |
| $(b+ab-(1)-a)/2n$ | B | 0.0181 | 1 | 0.0181 | 0.87 | 0.38 |
| $((1)+ab-a-b)/2n$ | AB | 0.0040 | 1 | 0.0040 | 0.19 | 0.67 |
| n=4 | Error | 0.2495 | 12 (4(4-1)) | 0.0208 | — | |
| | Total | 3.0695 | 15 (4n-1) | | | |

Conclusion: Only Factor A has a significant effect on outcome; has a + effect

# $2^k$ Factorial Design, $k \geq 3$

Table 12-16 Algebraic Signs for Calculating Effects in the $2^3$ Design

| Treatment Combination | Factorial Effect | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $I$ | $A$ | $B$ | $AB$ | $C$ | $AC$ | $BC$ | $ABC$ |
| (1) | + | − | − | + | − | + | + | − |
| $a$ | + | + | − | − | − | − | + | + |
| $b$ | + | − | + | − | − | + | − | + |
| $ab$ | + | + | + | + | − | − | − | − |
| $c$ | + | − | − | + | + | − | − | + |
| $ac$ | + | + | − | − | + | + | − | − |
| $bc$ | + | − | + | − | + | − | + | − |
| $abc$ | + | + | + | + | + | + | + | + |

$2^3$

| Treatment Combinations | Design Factors | | | Surface Roughness | Totals |
|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | | |
| (1) | −1 | −1 | −1 | 9, 7 | 16 |
| $a$ | 1 | −1 | −1 | 10, 12 | 22 |
| $b$ | −1 | 1 | −1 | 9, 11 | 20 |
| $ab$ | 1 | 1 | −1 | 12, 15 | 27 |
| $c$ | −1 | −1 | 1 | 11, 10 | 21 |
| $ac$ | 1 | −1 | 1 | 10, 13 | 23 |
| $bc$ | −1 | 1 | 1 | 10, 8 | 18 |
| $abc$ | 1 | 1 | 1 | 16, 14 | 30 |

- $SS_{A} = \dfrac{(\text{Contrast})^2}{n\,2^k}$ ; d.o.f = 1

  $B$, $C$, $AB$, $AC$, $BC, ABC$

  $\text{Effect} = \dfrac{\text{Contrast}}{n\,2^{k-1}}$

- $SS_E = 2^k(n-1)$ d.o.f.

- $SS_T = 2^k n - 1$ d.o.f.

$n$ = # replicates
$k$ = # factors

$a$ = Total response at that treatment level
$b$
$c$
$ab$
$ac$
$bc$
$abc$

Table 12-18 Analysis of Variance for the Surface Finish Experiment

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $f_0$ | $P$-value |
|---|---|---|---|---|---|
| $A$ | 45.5625 | 1 | 45.5625 | 18.69 | 0.0025 |
| $B$ | 10.5625 | 1 | 10.5625 | 4.33 | 0.0709 |
| $C$ | 3.0625 | 1 | 3.0625 | 1.26 | 0.2948 |
| $AB$ | 7.5625 | 1 | 7.5625 | 3.10 | 0.1162 |
| $AC$ | 0.0625 | 1 | 0.0625 | 0.03 | 0.8784 |
| $BC$ | 1.5625 | 1 | 1.5625 | 0.64 | 0.4548 |
| $ABC$ | 5.0625 | 1 | 5.0625 | 2.08 | 0.1875 |
| Error | 19.5000 | 8 | 2.4375 | | |
| Total | 92.9375 | 15 | | | |

# Multiple-Factor Analysis-Of-Variance

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

$i = 1 \cdots a$
$j = 1 \cdots b$
$k = 1 \cdots c$
$l = 1 \cdots n$

Table 12-10 Analysis of Variance Table for the Three-Factor Fixed Effects Model

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | Expected Mean Squares | $F_0$ |
|---|---|---|---|---|---|
| $A$ | $SS_A$ | $a-1$ | $MS_A$ | $\sigma^2 + \frac{bcn\sum\tau_i^2}{a-1}$ | $\frac{MS_A}{MS_E}$ |
| $B$ | $SS_B$ | $b-1$ | $MS_B$ | $\sigma^2 + \frac{acn\sum\beta_j^2}{b-1}$ | $\frac{MS_B}{MS_E}$ |
| $C$ | $SS_C$ | $c-1$ | $MS_C$ | $\sigma^2 + \frac{abn\sum\gamma_k^2}{c-1}$ | $\frac{MS_C}{MS_E}$ |
| $AB$ | $SS_{AB}$ | $(a-1)(b-1)$ | $MS_{AB}$ | $\sigma^2 + \frac{cn\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$ | $\frac{MS_{AB}}{MS_E}$ |
| $AC$ | $SS_{AC}$ | $(a-1)(c-1)$ | $MS_{AC}$ | $\sigma^2 + \frac{bn\sum\sum(\tau\gamma)_{ik}^2}{(a-1)(c-1)}$ | $\frac{MS_{AC}}{MS_E}$ |
| $BC$ | $SS_{BC}$ | $(b-1)(c-1)$ | $MS_{BC}$ | $\sigma^2 + \frac{an\sum\sum(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$ | $\frac{MS_{BC}}{MS_E}$ |
| $ABC$ | $SS_{ABC}$ | $(a-1)(b-1)(c-1)$ | $MS_{ABC}$ | $\sigma^2 + \frac{n\sum\sum\sum(\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$ | $\frac{MS_{ABC}}{MS_E}$ |
| Error | $SS_E$ | $abc(n-1)$ | $MS_E$ | $\sigma^2$ | |
| Total | $SS_T$ | $abcn-1$ | | | |

$(abcn-1)$ $\quad SS_T = \sum_i^a \sum_j^b \sum_k^c \sum_l^n y_{ijkl}^2 - \frac{y_{....}^2}{abcn}$

$(a-1)$ $\quad SS_A = \sum_i^a \frac{y_{i...}^2}{bcn} - \frac{y_{....}^2}{abcn}$

$(b-1)$ $\quad SS_b = \sum_j^b \frac{y_{.j..}^2}{acn} - \frac{y_{....}^2}{abcn}$

$(c-1)$ $\quad SS_c = \sum_k^c \frac{y_{..k.}^2}{abn} - \frac{y_{....}^2}{abcn}$

$(a-1)(b-1)$ $\quad SS_{AB} = \sum_{i}^{a}\sum_{j}^{b} \frac{y_{ij\cdot\cdot}^2}{cn} - \frac{y_{\cdot\cdot\cdot\cdot}^2}{abcn} - SS_A - SS_B$

$(a-1)(c-1)$ $\quad SS_{AC} = \sum_{i}^{a}\sum_{k}^{c} \frac{y_{i\cdot k\cdot}^2}{bn} - \frac{y_{\cdot\cdot\cdot\cdot}^2}{abcn} - SS_A - SS_C$

$(b-1)(c-1)$ $\quad SS_{BC} =$

$(a-1)(b-1)(c-1)$ $\quad SS_{ABC} = \sum_{i}^{a}\sum_{j}^{b}\sum_{k=1}^{c} y_{ijk\cdot}^2 - \frac{y_{\cdot\cdot\cdot\cdot}^2}{abcn} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$

$abc(n-1)$ $\quad SS_E = SS_T - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC} - SS_{ABC}$

$$F_i = \frac{MS_i}{MS_E} \qquad i = a, b, c, ab, ac, bc, abc$$

- $S^2$ or MSE is an unbiased estimator for $\sigma^2$.

Test for a Significant Effects

## Questions ANOVA can answer:

- Is the A factor significant?
  B
  C
  ⋮
- Is the interaction effect of AB (AC, BC, ...) significant?
- Is the ABC, ABD, BCD, .... effect significant?
- Is ABCD, ---- effect significant?