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Course Lecture Notes

SIXTEEN

SECTION 3.6 DIFFERENTIAL EQUATIONS OF CONTINUITY
SECTION 3.7 DIFFERENTIAL EQUATIONS OF MOTION Geankoplis page 164-174

We have solved systems with material and mechanical energy balances on macroscopic control volumes. We have also used shell balances, which use a control volume with a differential dimension in one direction, to obtain velocity and stress profiles in that direction. To advance our study, we now need to consider a control volume with differential elements in all three spatial dimensions.

A. Goals:

- Derive general differential equations of continuity (mass balance).
- Derive general differential equation of change (momentum balance).
- Use these generalized equations to solve a particular problem by keeping only terms relevant to the problem while discarding unnecessary terms.

B. Types of derivatives:

1. partial time derivative

$$\frac{\partial \rho}{\partial t} = \text{change in density at a fixed point (x,y,z) with time}$$

We are in a canoe on a river, paddling with just enough effort so that we don't move at all. Then we look down and see $\frac{\partial \rho}{\partial t}$ at the same point in the river. (Fixed position, different particles (elements) of fluid.)

2. total time derivative

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$\frac{d\rho}{dt} = \text{change in density of the fluid while we move around with some velocity}$$

$$\underline{v} = \begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \end{bmatrix}^T$$

We are in a canoe on a river, paddling around, maybe cross-stream, maybe upstream, with a given velocity, $\underline{v}$, a vector. Then we look down and see $\frac{\partial \rho}{\partial t}$ at different points in river and different particles of the fluid. (Different position, different particles (elements) of fluid.)

3. Substantial time derivative (a subset of total time derivatives)

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} v_x + \frac{\partial \rho}{\partial y} v_y + \frac{\partial \rho}{\partial z} v_z = \frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho$$

$\frac{D\rho}{Dt} =$ =change in density of the fluid while moving with the fluid velocity

$$\underline{v} = [v_x \quad v_y \quad v_z]^T$$

We are in a canoe on a river, allowing ourselves to move with the river (no paddling). Then we look down and see $\frac{\partial \rho}{\partial t}$ at different point in river but for the same particle of the fluid. (Different position, same particles (elements) of fluid.)

C. Linear Algebra and Vector Calculus Operations:

1. Gradient of a scalar

$$\nabla \rho = \begin{bmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \end{bmatrix}^T$$

The gradient of a scalar is a 3x1 vector of the derivative of that scalar in all spatial dimensions.

2. divergence of a vector

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The divergence of a vector is a scalar, representing the dot product of the gradient operator and the vector.

3. divergence of a matrix

$$\nabla \cdot \underline{\underline{\tau}} = \begin{bmatrix} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{bmatrix}$$

The divergence of a 3x3 matrix is a 3x1 vector, representing the dot product of the gradient operator and the matrix.

4. Laplacian of a scalar

$$\nabla^2 \rho = \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}$$

This is a scalar. The Laplacian is the sequential operation of first the gradient operator and then the divergence operator.

Check out some rules of vector algebra and vector calculus in equations (3.6-4 to equation 3.6-16), Geankoplis, page 166-167.

D. Derive the general differential equation of continuity

accumulation = in - out + generation - consumption

Define differential volume element as shown on page167, Geankoplis.
Define the five terms in the mass balance.
There is no generation or consumption of the fluid.

$$\text{gen} = \text{con} = 0$$

$$\text{acc} = V\frac{\partial \rho}{\partial t} = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

$$\text{in} = \text{in}_x + \text{in}_y + \text{in}_z = A_{y-z}\rho v_x\big|_x + A_{x-z}\rho v_y\big|_y + A_{x-y}\rho v_z\big|_z$$
$$= \Delta y \Delta z \rho v_x\big|_x + \Delta x \Delta z \rho v_y\big|_y + \Delta x \Delta y \rho v_z\big|_z$$

$$\text{out} = \Delta y \Delta z \rho v_x\big|_{x+\Delta x} + \Delta x \Delta z \rho v_y\big|_{y+\Delta y} + \Delta x \Delta y \rho v_z\big|_{z+\Delta z}$$

Put these five terms in mass balance:

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \left(\Delta y \Delta z \rho v_x\big|_x + \Delta x \Delta z \rho v_y\big|_y + \Delta x \Delta y \rho v_z\big|_z\right)$$
$$- \left(\Delta y \Delta z \rho v_x\big|_{x+\Delta x} + \Delta x \Delta z \rho v_y\big|_{y+\Delta y} + \Delta x \Delta y \rho v_z\big|_{z+\Delta z}\right)$$

Divide by differential volume:

$$\frac{\partial \rho}{\partial t} = \left(\frac{\rho v_x\big|_x}{\Delta x} + \frac{\rho v_y\big|_y}{\Delta y} + \frac{\rho v_z\big|_z}{\Delta z}\right)$$
$$- \left(\frac{\rho v_x\big|_{x+\Delta x}}{\Delta x} + \frac{\rho v_y\big|_{y+\Delta y}}{\Delta y} + \frac{\rho v_z\big|_{z+\Delta z}}{\Delta z}\right)$$

Rearrange into a form recognizable as the definition of a derivative:

$$-\frac{\partial \rho}{\partial t} = \frac{\rho v_x\big|_{x+\Delta x} - \rho v_x\big|_x}{\Delta x} + \frac{\rho v_y\big|_{y+\Delta y} - \rho v_y\big|_y}{\Delta y} + \frac{\rho v_z\big|_{z+\Delta z} - \rho v_z\big|_z}{\Delta z}$$

Take limits as differential elements approach 0 and apply the definition of the derivative:

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = \nabla \cdot \rho \underline{v}$$

Now consider the law for the derivative

$$\frac{\partial(\rho v_x)}{\partial x} = \rho \frac{\partial(v_x)}{\partial x} + v_x \frac{\partial(\rho)}{\partial x}$$ and the same for y and z

$$-\frac{D\rho}{Dt} = \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

$$-\frac{D\rho}{Dt} = \rho(\nabla \cdot \underline{v})$$

This is the continuity equation.

For the case of an incompressible fluid (constant density) at steady or unsteady state:

$$\nabla \cdot \underline{v} = 0$$

This is a mass balance. It may not look like it but it is. Just go back through the derivation and see that this is nothing but an expression of

accumulation = in - out + generation - consumption

when there is no generation or consumption and when the fluid is incompressible. This equation does not assume steady state, even though there is no time derivative in the equation. This is a first order partial differential equation PDE)

Example 3.6-1. page 168

The continuity equation can also be expressed in spherical and cylindrical coordinates, which are useful if you have a system which naturally lends itself to that system, as a circular pipe lends itself to cylindrical coordinates.

In cylindrical coordinates:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \rho \underline{v}$$ is still true but

$x = r\cos\theta$, $y = r\sin\theta$, and $z = z$

so the continuity equation becomes:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \rho \underline{v} = \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}$$

In spherical coordinates:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \rho \underline{v}$$ is still true but

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$ SO:

so the continuity equation becomes:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \rho \underline{v} = \frac{1}{r^2}\frac{\partial\left(\rho r^2 v_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\rho v_\theta \sin\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial\left(\rho v_\phi\right)}{\partial \phi}$$

Section 3.7 Now do the same analysis for a momentum balance

E. Derive the general differential equation of change (momentum balance)

accumulation = in - out + generation - consumption

Define differential volume element as shown on page 167.

Momentum is a vector. Since there is no intrinsic difference between x, y, and z coordinates, we can derive the equation of change for the x-component of momentum and then make analogous statements about the y and z components. Look only at x-component of momentum

$$acc = V\frac{\partial\left(\rho v_x\right)}{\partial t} = \Delta x \Delta y \Delta z \frac{\partial\left(\rho v_x\right)}{\partial t}$$

Momentum can flow in and out by convection:

$$in = in_x + in_y + in_z = A_{y-z}\rho v_x v_x \,|_x + A_{x-z}\rho v_x v_y \,|_y + A_{x-y}\rho v_x v_z \,|_z$$
$$= \Delta y \Delta z \rho v_x v_x \,|_x + \Delta x \Delta z \rho v_x v_y \,|_y + \Delta x \Delta y \rho v_x v_z \,|_z$$

$$out = \Delta y \Delta z \rho v_x v_x \,|_{x+\Delta x} + \Delta x \Delta z \rho v_x v_y \,|_{y+\Delta y} + \Delta x \Delta y \rho v_x v_z \,|_{z+\Delta z}$$

Momentum can flow in and out by molecular diffusion:

$$in = in_x + in_y + in_z = A_{y-z}\tau_{xx} \,|_x + A_{x-z}\tau_{yx} \,|_y + A_{x-y}\rho\tau_{zx} \,|_z$$
$$\Delta y \Delta z \tau_{xx} \,|_x + \Delta x \Delta z \tau_{yx} \,|_y + \Delta x \Delta y \tau_{zx} \,|_z$$

$$out = \Delta y \Delta z \tau_{xx} \,|_{x+\Delta x} + \Delta x \Delta z \tau_{yx} \,|_{y+\Delta y} + \Delta x \Delta y \tau_{zx} \,|_{z+\Delta z}$$

Momentum can be generated in the differential volume by a body force like gravity:

$$gen = V\rho g_x = \Delta x \Delta y \Delta z \rho g_x$$

Momentum can be generated in the differential volume by a net force acting on the element, due to the difference in pressures:

$$gen = (p_x - p_{x+\Delta x})\Delta y \Delta z$$

Put these five terms in mass balance:

$$\begin{aligned}
\Delta x \Delta y \Delta z \frac{\partial(\rho v_x)}{\partial t} &= \Delta y \Delta z \rho v_x v_x \,|_x + \Delta x \Delta z \rho v_x v_y \,|_y + \Delta x \Delta y \rho v_x v_z \,|_z \\
&- \Delta y \Delta z \rho v_x v_x \,|_{x+\Delta x} + \Delta x \Delta z \rho v_x v_y \,|_{y+\Delta y} + \Delta x \Delta y \rho v_x v_z \,|_{z+\Delta z} \\
&+ \Delta y \Delta z \tau_{xx} \,|_x + \Delta x \Delta z \tau_{yx} \,|_y + \Delta x \Delta y \tau_{zx} \,|_z \\
&- \Delta y \Delta z \tau_{xx} \,|_{x+\Delta x} + \Delta x \Delta z \tau_{yx} \,|_{y+\Delta y} + \Delta x \Delta y \tau_{zx} \,|_{z+\Delta z} \\
&+ \Delta x \Delta y \Delta z \rho g_x + (p_x - p_{x+\Delta x})\Delta y \Delta z
\end{aligned}$$

Divide by volume, rearrange, and take limits and apply the definition of the derivative

$$\begin{aligned}
-\frac{\partial(\rho v_x)}{\partial t} &= \left[\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z}\right] \\
&+ \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \frac{\partial p}{\partial x} - \rho g_x
\end{aligned}$$

Now, consider again the rule for derivatives of a product:

$$\frac{\partial(\rho v_x v_x)}{\partial x} = \rho v_x \frac{\partial(v_x)}{\partial x} + v_x \frac{\partial(\rho v_x)}{\partial x}$$

and the same for y and z, i.e.

$$\frac{\partial(\rho v_x v_y)}{\partial y} = \rho v_x \frac{\partial(v_y)}{\partial y} + v_y \frac{\partial(\rho v_x)}{\partial y}$$

$$\frac{\partial(\rho v_x v_z)}{\partial z} = \rho v_x \frac{\partial(v_z)}{\partial z} + v_z \frac{\partial(\rho v_x)}{\partial z}$$

Substituting these rules into our momentum balance, we have:

$$-\frac{\partial(\rho v_x)}{\partial t}=\begin{bmatrix}\rho v_x\frac{\partial(v_x)}{\partial x}+v_x\frac{\partial(\rho v_x)}{\partial x}\\+\rho v_x\frac{\partial(v_y)}{\partial y}+v_y\frac{\partial(\rho v_x)}{\partial y}\\+\rho v_x\frac{\partial(v_z)}{\partial z}+v_z\frac{\partial(\rho v_x)}{\partial z}\end{bmatrix}$$

$$+\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\frac{\partial p}{\partial x}-\rho g_x$$

Recall, the substantial derivative of the x-momentum component has the definition:

$$\frac{D(\rho v_x)}{Dt}=\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}v_x+\frac{\partial(\rho v_x)}{\partial y}v_y+\frac{\partial(\rho v_x)}{\partial z}v_z$$

$$-\left[\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}v_x+\frac{\partial(\rho v_x)}{\partial y}v_y+\frac{\partial(\rho v_x)}{\partial z}v_z\right]=$$

$$\left[\rho v_x\frac{\partial(v_x)}{\partial x}+\rho v_x\frac{\partial(v_y)}{\partial y}+\rho v_x\frac{\partial(v_z)}{\partial z}\right]+\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\frac{\partial p}{\partial x}-\rho g_x$$

$$-\frac{D(\rho v_x)}{Dt}=\rho v_x(\nabla\cdot\underline{v})+\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\frac{\partial p}{\partial x}-\rho g_x$$

but using the product rule for differentiation again:

$$\frac{D(\rho v_x)}{Dt}=\rho\frac{D(v_x)}{Dt}+v_x\frac{D(\rho)}{Dt}$$

From the mass balance: $-\frac{D\rho}{Dt}=\rho(\nabla\cdot\underline{v})$

so $\frac{D(\rho v_x)}{Dt}=\rho\frac{D(v_x)}{Dt}+v_x\frac{D(\rho)}{Dt}$

$$\frac{D(\rho v_x)}{Dt}=\rho\frac{D(v_x)}{Dt}-v_x\rho(\nabla\cdot\underline{v})$$

And the momentum balance becomes:

$$-\rho\frac{D(v_x)}{Dt}+v_x\rho(\nabla\cdot\underline{v})-\rho v_x(\nabla\cdot\underline{v})=\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\frac{\partial p}{\partial x}-\rho g_x$$

$$-\rho\frac{D(v_x)}{Dt}=\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+\frac{\partial p}{\partial x}-\rho g_x$$

This is the equation of motion in the x-direction. It is a momentum balance.
The y- and z-components of the momentum are obtained in a precisely analogous manner.
They look like:

$$-\rho\frac{D(v_y)}{Dt}=\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+\frac{\partial p}{\partial y}-\rho g_y$$

$$-\rho\frac{D(v_z)}{Dt}=\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+\frac{\partial p}{\partial z}-\rho g_z$$

which in vector notation looks like:

$$-\rho\frac{D(\underline{v})}{Dt}=\nabla\cdot\underline{\underline{\tau}}+\nabla p-\rho\underline{g}$$ Geankoplis, 3.7-13

This momentum balance is true for any continuous medium.

B. Equations for stresses using Newton's Law of viscosity (page 172-173)

1. Shear stress components for Newtonian fluids in rectangular coordinates
   See Geankoplis page 172, eqns (3.7-14 to 3.7-20)
2. Shear stress components for Newtonian fluids in cylindrical coordinates
   See Geankoplis page 172-173, eqns (3.7-21 to 3.7-27)
3. Shear stress components for Newtonian fluids in spherical coordinates
   See Geankoplis page 173, eqns (3.7-28 to 3.7-34)

For the qualitative explanation of the basis of these equations, you must go to "An Introduction to Fluid Dynamics" by Stanley Middleman, Wiley, New York, 1998, page 142-143

Examine rectangular case: obtain 1-D results from 3-D case by looking at equation 3.7-17 when the y component of the velocity is zero.

Substitute the definitions of the Newtonian stress into the equation of change. Obtain the equation of change for a Newtonian fluid. Example given for x component in rectangular coordinate system. (3.7-35)

$$\rho\frac{D(v_x)}{Dt} = \frac{\partial}{\partial x}\left[2\mu\frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\nabla\cdot\underline{v})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x$$

C. Navier-Stokes equations (Equation of change for incompressible Newtonian fluid) (page 174-175)

When the density is constant (incompressible) and the viscosity is constant (isothermal conditions), the equations of (3.7-10 to 3.7-13) combined with Newton's law equations (equations 3.7-14 to 3.7-34) become the Navier-Stokes equations.

1. Equations of change for incompressible Newtonian fluid in rectangular coordinates

Start with equation (3.7-35), general equation of change for Newtonian fluid and assume constant density:

$$\rho\frac{D(v_x)}{Dt} = \frac{\partial}{\partial x}\left[2\mu\frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\nabla\cdot\underline{v})\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho\frac{D(v_x)}{Dt} = \left[2\mu\frac{\partial^2 v_x}{\partial x^2} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\nabla\cdot\underline{v})\right] + \left[\mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_y}{\partial y\partial x}\right)\right] + \left[\mu\left(\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z\partial x}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho\frac{D(v_x)}{Dt} = \left[2\mu\frac{\partial^2 v_x}{\partial x^2} - \frac{2}{3}\mu\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\right] + \left[\mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_y}{\partial y\partial x}\right)\right] + \left[\mu\left(\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z\partial x}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho \frac{D(v_x)}{Dt} = \left[2\mu \frac{\partial^2 v_x}{\partial x^2} - \frac{2}{3}\mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z} \right) \right]$$
$$+ \left[\mu \left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_y}{\partial y \partial x} \right) \right] + \left[\mu \left(\frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial z \partial x} \right) \right] - \frac{\partial p}{\partial x} + \rho g_x$$

$$\rho \frac{D(v_x)}{Dt} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x$$
$$+ \frac{\mu}{3} \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z} \right]$$

$$\rho \frac{D(v_x)}{Dt} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x$$
$$+ \frac{\mu}{3} \frac{\partial}{\partial x} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]$$

$$\rho \frac{D(v_x)}{Dt} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x + \frac{\mu}{3} \frac{\partial}{\partial x} [\nabla \cdot \underline{v}]$$

From the mass balance, we know that for an incompressible fluid, the divergence of the velocity is zero.

$$\rho \frac{D(v_x)}{Dt} = \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

This is the equation of change for an incompressible Newtonian fluid in the x-direction, using rectangular coordinates. The y and z equations look like:

$$\rho \frac{D(v_y)}{Dt} = \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) - \frac{\partial p}{\partial y} + \rho g_y$$

$$\rho\frac{D(v_z)}{Dt} = \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - \frac{\partial p}{\partial z} + \rho g_z$$

and the three equations can be expressed in vector notation as

$$\rho\frac{D(\underline{v})}{Dt} = \mu\nabla^2\underline{v} - \nabla p + \rho\underline{g}$$

2. Equations of change for incompressible Newtonian fluids in cylindrical coordinates
   See Geankoplis page 174, eqns (3.7-40 to 3.7-42)
3. Equations of change for incompressible Newtonian fluids in spherical coordinates
   See Geankoplis page 174-175, eqns (3.7-43 to 3.7-46)